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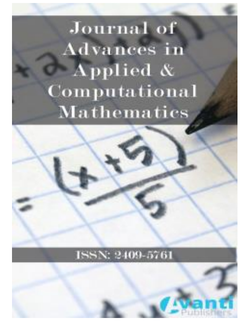

Generalized Rational Type Contraction and Fixed Point Theorems in Partially Ordered Metric Spaces

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ABSTRACT

In this article, we establish the existence and uniqueness of fixed points for rational type contraction mappings in a metric space that is equipped with a partial order. Our results are shown to improve upon previous results in the literature, and we provide illustrative examples to demonstrate the effectiveness of our approach.

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1. Introduction

M. Frechet's introduction of metric spaces in 1906 was a ground breaking development in the field of functional analysis. This new type of space provided a natural setting for studying functions and operators, and has since been generalized and expanded upon in a variety of ways. Several researchers have developed generalized metric spaces, such as complex valued metric spaces, rectangular metric spaces, semi metric spaces, quasimetric spaces, and more. These spaces have found applications in a wide range of areas, including probability, computer science, and functional analysis.

There are numerous results in the literature that extend or improve upon the existence and uniqueness of fixed points. One such result is that of Ran and Reurings [1], who studied fixed points for certain mappings in partially ordered metric spaces. Nieto and Lopez [2] later generalized this result for non-decreasing mappings and demonstrated its applications to partial differential equations.

The mixed monotone property of contractive operators was introduced by Bhaskar and Lakshmikantham [3], who considered the case of partially ordered metric spaces. They then derived several coupled fixed point theorems using this property. Chatterji [4] later considered contractive conditions for self-mappings in metric spaces, and Dass and Gupta [5] studied rational type contractions in complete metric spaces.

Recently, Seshagiri Rao and Kalyani [6-8] have explored some results on fixed point, coupled fixed point and coincidence point for the mappings in partially ordered metric spaces satisfying rational type contraction.

Motivated by the research of Seshagiri Rao and Kalyani [9], we introduce a class of mappings known as generalized rational type contractive mappings and explore the properties of these mappings and derive fixed point results for them in partially ordered metric spaces. We also provide illustrative examples to demonstrate the improvements offered by our approach.

2. Preliminaries

In this section, we start with the following definitions and theorems that motivate our study as follows:

Definition 2.1. [10] The triple $(X, d, \preccurlyeq)$ is called partially ordered metric spaces, if $(X, \preccurlyeq)$ is a partially ordered set and (X, d) is a metric space.

Definition 2.2. [11] Let (X, d) be a complete metric space, then the triple $(X, d, \preccurlyeq)$ is called complete partially ordered metric spaces.

Definition 2.3. [12] A partially ordered metric space $(X, d, \preccurlyeq)$ is called ordered complete if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$, the following condition holds: either

i. if x_n is a non-increasing sequence in X such that $x_n \to x$ implies $x \preccurlyeq x_n$, for all $n \in \mathbb{N}$ that is, $x = \inf\{x_n\}$, or

ii. if x_n is a non-decreasing sequence in X such that $x_n \to x$ implies $x_n \preccurlyeq x$, for all $n \in \mathbb{N}$ that is, $x = \sup\{x_n\}$.

Definition 2.4. [13] Let $(X, \preccurlyeq)$ be a partially ordered set and let $T: X \to X$ be a mapping. Then

i. elements $x, y \in X$ are comparable, if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds;

ii. a non-empty set X is called well ordered set, if every two elements of it are comparable;

iii. T is said to be monotone non-decreasing with respect to $\preccurlyeq$, if for all $x, y \in X$,

$$x \preccurlyeq y \text{ implies } Tx \preccurlyeq Ty.$$

iv. T is said to be monotone non-increasing with respect to $\preccurlyeq$, if for all $x, y \in X$,

$$x \preccurlyeq y \text{ implies } Tx \succcurlyeq Ty.$$

Theorem 2.4. [7] Let $T: X \to X$ be a complete metric space (X, d) satisfying the following condition

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)[1 + d(y, Ty)]}{1 + d(x, y)} + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] + \delta d(x, y) \tag{2.1}$$

for all distinct $x, y \in X$, where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + 2(\beta + \gamma) + \delta < 1$. Then T has a unique fixed point in X.

For more details on generalized metric spaces, one can see [14-24].

3. Main Results

Theorem 3.1. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that $T: X \to X$ is a non-decreasing, continuous self mapping satisfying

$$d(Tx, Ty) \leq \begin{cases} \varepsilon d(x, y) + \delta[d(x, Ty) + d(y, Tx)] + \gamma \dfrac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta \dfrac{d(x, Tx)d(y, Ty)}{d(x, y) + d(x, Ty) + d(y, Tx)} \\ \quad + \alpha \dfrac{d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)}{d(y, Tx) + d(x, Ty)}, \qquad if\ K \neq 0 \\ 0 \qquad , if\ K = 0 \end{cases} \tag{3.1}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, where $K = d(y, Tx) + d(x, Ty)$ and $\alpha, \beta, \gamma, \delta, \varepsilon \geq 0$ with $\varepsilon + 2\delta + \gamma + \beta + \alpha < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

Proof. Suppose $x_0 = Tx_0$, then we have the result complete. Let that $x_0 \prec Tx_0$. Since T is a non-decreasing mapping, then by induction we obtain that

$$x_0 \prec Tx_0 \preccurlyeq T^2x_0 \preccurlyeq \cdots \preccurlyeq T^n x_0 \preccurlyeq T^{n+1}x_0 \preccurlyeq \tag{3.2}$$

Now, we construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. Since T is nondecreasing mapping, we get

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq$$

Suppose there exists $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then $x_{n+1} = Tx_n = x_n$, implies x_n is a fixed point and the proof is finished. Again, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since the points x_n and x_{n-1} are comparable for $n \in \mathbb{N}$ with (3.2), we have the following two cases:

Case 1: If $K = d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) \neq 0$, then, with (3.1), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \varepsilon d(x_n, x_{n-1}) + \delta[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] + \gamma \frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})} \\ &+ \beta \frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1}) + d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)} + \alpha \frac{d(x_n, Tx_n)d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})} \end{aligned}$$

implies that

$$\begin{aligned} d(x_{n+1}, x_n) \leq \varepsilon d(x_n, x_{n-1}) + \delta[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] + \gamma \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_n, x_{n-1})} \\ + \beta \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_n, x_{n-1}) + d(x_n, x_n) + d(x_{n-1}, x_{n+1})} + \alpha \frac{d(x_n, x_{n+1})d(x_n, x_n) + d(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)}{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)} \end{aligned}$$

By triangular inequality $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+1})$ and $d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$, we have

$$d(x_{n+1},x_n) \le \left(\frac{\varepsilon+\delta+\beta+\alpha}{1-\delta-\gamma}\right) d(x_{n-1},x_n)$$

Inductively, we have

$$d(x_{n+1},x_n) \le r^n d(x_0,x_1)$$

where $r = \frac{\varepsilon+\delta+\beta+\alpha}{1-\delta-\gamma} < 1$. Now, we prove that $\{x_n\}$ is a Cauchy sequence. For $m \ge n$ and by triangular inequality, we have

$$d(x_m,x_n) \le d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + \cdots + d(x_{n+1},x_n) \le \frac{r^n}{1-r} d(x_1,x_0)$$

as $m,n \to +\infty$, $d(x_m,x_n) \to 0$. Thus, $\{x_n\}$ is a Cauchy sequence in a complete metric space X. Hence, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Then, by continuity of T we have

$$\begin{aligned} Tu &= T\left(\lim_{n\to\infty} x_n\right) \\ &= \lim_{n\to\infty} Tx_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= u. \end{aligned}$$

Thus, u is a fixed point of T.

Case 2: If $K = d(x_{n-1},Tx_n) + d(x_n,Tx_{n-1}) = 0$, then $d(x_{n+1},x_n) = 0$. Which implies that $x_{n+1} = x_n$, a contradiction as the sequence points is comparable. Thus, there exists a fixed point u of T.

Example 3.2. Suppose $X = \{(1,0),(0,1)\} \subseteq \mathbb{R}^2$, and define the usual order

$$U: (u,v) \preccurlyeq (z,t) \Leftrightarrow u \preccurlyeq z \text{ and } v \preccurlyeq t. \tag{3.3}$$

Let $T: X \to X$ define by $T(x,y) = (x,y)$. Then T has a fixed point in X.

Proof. Let $(X,\preccurlyeq)$ be a partially ordered set, whose different elements are not comparable. Besides, (X,d_2) is a complete metric space considering d_2, the Euclidean distance. The identity map $T(x,y) = (x,y)$ is trivially continuous and nondecreasing and condition

$$\begin{aligned} & d\big(T(u,v),T(z,t)\big) \\ & \le \varepsilon d\big((u,v),(z,t)\big) + \delta\big[d\big((u,v),T(z,t)\big) + d\big((z,t),T(u,v)\big)\big] + \gamma\frac{d\big((u,v),T(u,v)\big)d\big((z,t),T(z,t)\big)}{d\big((u,v),(z,t)\big)} \\ & + \beta\frac{d\big((u,v),T(u,v)\big)d\big((z,t),T(z,t)\big)}{d\big((u,v),(z,t)\big) + d\big((u,v),T(z,t)\big) + d\big((z,t),T(u,v)\big)} \\ & + \alpha\frac{d\big((u,v),T(u,v)\big)d\big((u,v),T(z,t)\big) + d\big((z,t),T(u,v)\big)d\big((z,t),T(z,t)\big)}{d\big((z,t),T(u,v)\big) + d\big((u,v),T(z,t)\big)} \end{aligned}$$

for all $\alpha,\beta,\gamma,\delta,\varepsilon \in [0,1)$ with $\varepsilon + 2\delta + \gamma + \beta + \alpha < 1$. Since elements in X are only comparable to themselves. Furthermore, $(1,0) \preccurlyeq T(1,0) = (1,0)$. Thus, there are two fixed points in X, the assertions in Theorem 3.1 holds and T has two fixed points, $(1,0)$ and $(0,1)$.

Now, we prove that Theorem 3.1 is still valid for T, not necessarily continuous, by assuming the following hypothesis in X.

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then

$$x = sup\{x_n\}. \tag{3.4}$$

Theorem 3.3. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Assume that $T: X \to X$ is a monotone non-decreasing self mapping satisfying

$$d(Tx,Ty) \leq \begin{cases} \varepsilon d(x,y) + \delta[d(x,Ty) + d(y,Tx)] + \gamma \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta \dfrac{d(x,Tx)d(y,Ty)}{d(x,y) + d(x,Ty) + d(y,Tx)} \\ +\alpha \dfrac{d(x,Tx)d(x,Ty) + d(y,Tx)d(y,Ty)}{d(y,Tx) + d(x,Ty)}, \qquad if\ K \neq 0 \\ 0 \qquad ,if\ K = 0 \end{cases} \tag{3.5}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, where $K = d(y,Tx) + d(x,Ty)$ and $\alpha, \beta, \gamma, \delta, \varepsilon \geq 0$ with $\varepsilon + 2\delta + \gamma + \beta + \alpha < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

Proof. In view of the proof of Theorem 3.1, we have $\{x_n\}$ is Cauchy sequence. Now, we need to check that $Tu = u$. Since $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to u$, then $u = sup\{x_n\}$ for all $n \in \mathbb{N}$ by (3.4). Also since T is a non-decreasing mapping, then $Tx_n \preccurlyeq Tu$ for all $n \in \mathbb{N}$ or, equivalently, $x_{n+1} \preccurlyeq Tu$ for all $n \in \mathbb{N}$. Moreover, as $x_0 \prec x_1 \preccurlyeq Tu$ and $u = sup\{x_n\}$, we get $u \preccurlyeq Tu$. Suppose that $u \prec Tu$. Using the same argument as that in the proof of Theorem 3.1 for $x_0 \preccurlyeq Tx_0$, we get a non-decreasing sequence $\{T^n u\}$ in X such that $\lim_{n\to+\infty} T^n u = y$ for certain $y \in X$. Again, with (3.4), we obtain that $y = sup\{T^n u\}$. Moreover, from $x_0 \preccurlyeq u$, we get $x_n \preccurlyeq T^n x_0 \preccurlyeq T^n u$ for $n \geq 1$ and $x_n \prec T^n u$ for $n \geq 1$ because $x_n \preccurlyeq u \prec Tu \preccurlyeq T^n u$ for all $n \geq 1$.

Since x_n and $T^n u$ are comparable and distinct for $n \geq 1$, we consider the following cases:

Case 1: If $d(T^n u, Tx_n) + d(x_n, T^{n+1}u) \neq 0$, then, by using (3.1), we have

$$\begin{aligned} d(x_{n+1}, T^{n+1}u) &= d\big(Tx_n, T(T^n u)\big) \\ &\leq \varepsilon d(x_n, T^n u) + \delta[d(x_n, T^{n+1}u) + d(T^n u, x_{n+1})] + \gamma \frac{d(x_n,\ x_{n+1})d(T^n u, T^{n+1}u)}{d(x_n, T^n u)} \\ &+ \beta \frac{d(x_n,\ x_{n+1})d(T^n u, T^{n+1}u)}{d(x_n, T^n u) + d(x_n, T^{n+1}u) + d(T^n u, x_{n+1})} + \alpha \frac{d(x_n,\ x_{n+1})d(x_n, T^{n+1}u) + d(T^n u, x_{n+1})d(T^n u, T^{n+1}u)}{d(T^n u,\ x_{n+1}) + d(x_n, T^{n+1}u)} \end{aligned}$$

On letting $n \to +\infty$, we have

$$\begin{aligned} d(u,y) \leq \varepsilon d(u,y) + \delta[d(u,y) + d(y,u)] + \gamma \frac{d(u,u)d(y,y)}{d(u,y)} + \beta \frac{d(u,u)d(y,y)}{d(u,y) + d(u,y) + d(y,z)} \\ + \alpha \frac{d(u,u)d(u,y) + d(y,u)d(y,y)}{d(y,u) + d(u,y)} \end{aligned}$$

$$d(u,y) \leq (\varepsilon + 2\delta)d(u,y)$$

as $\varepsilon + 2\delta < 1$, $d(u,y) = 0$, that is, $u = y$. Hence, $u = y = sup\{T^n u\}$ and consequently, $Tu \preccurlyeq u$, a contradiction. Thus, u is a fixed point of T.

Case 2: If $d(T^n u, Tx_n) + d(x_n, T^{n+1}u) = 0$, then $d(x_{n+1}, T^{n+1}u) = 0$. Taking the limit as $n \to +\infty$, we get $d(u,y) = 0$. That is, $u = y = sup\{T^n u\}$, which implies that $Tu \preccurlyeq u$, a contradiction. Thus, $Tu = u$.

To show uniqueness of fixed point that exists in Theorem 3.1 and Theorem 3.3, we give sufficient condition as follows

every pair of elements has a lower bound or an upper bound. (3.6)

In [12], the above condition is equivalent to for every $y, z \in X$, there exists $x \in X$ that is comparable to y and z.

Theorem 3.4. In addition to the hypothesis of Theorem 3.1 (or Theorem 3.3), assume that for every $y, z \in X$, there exists $x \in X$ that is comparable to y and z, then T has a unique fixed point.

Proof. In view of Theorem 3.1 (or Theorem 3.3), the set of fixed points of T is non-empty. Let $y, z \in X$ be two fixed points of T. Then, we distinguish two cases:

Case 1: If y and z are comparable and $y \neq z$. Then, we obtain the following two subcases:

(i). If $d(z,Ty) + d(y,Tz) \neq 0$, then using (3.1), we have

$$\begin{aligned} d(y,z) &= d(Ty,Tz) \\ &\leq \varepsilon d(y,z) + \delta[d(y,Tz) + d(z,Ty)] + \gamma \frac{d(y,Ty)d(z,Tz)}{d(y,z)} + \beta \frac{d(y,Ty)d(z,Tz)}{d(y,z) + d(y,Tz) + d(z,Ty)} \\ &\quad + \alpha \frac{d(y,Ty)d(y,Tz) + d(z,Ty)d(z,Tz)}{d(z,Ty) + d(y,Tz)} \\ &= \varepsilon d(y,z) + \delta[d(y,z) + d(z,y)] + \gamma \frac{d(y,y)d(z,z)}{d(y,z)} + \beta \frac{d(y,y)d(z,z)}{d(y,z) + d(y,z) + d(z,y)} \\ &\quad + \alpha \frac{d(y,y)d(y,z) + d(z,y)d(z,z)}{d(z,y) + d(y,z)} \end{aligned}$$

implies

$$\begin{aligned} d(y,z) &\leq (\varepsilon + 2\delta) d(y,z) \\ &< d(y,z) \text{ as } \varepsilon + 2\delta < 1 \end{aligned}$$

a contradiction. Thus, $y = z$.

(ii). If $d(z,Ty) + d(y,Tz) = 0$, then $d(y,z) = 0$, a contradiction again. Hence, $y = z$.

Case 2: If y and z are not comparable, then, by (3.1), there exists $x \in X$ comparable to y and z. By monotonicity it implies that $T^n x$ is comparable to $T^n y = y$ and $T^n z = z$ for $n = 0, 1, 2, \dots$.

If there exists $n_0 \geq 1$ such that $T^{n_0} x = y$, then as y is a fixed point, the sequence $\{T^n x : n \geq n_0\}$ is constant and consequently $\lim\limits_{n \to +\infty} T^n x = y$. On the other hand, if $T^n x \neq y$ for all $n \geq 1$.

Again, we have two subcases:

(i). If $d(T^{n-1}y, T^n x) + d(T^{n-1}x, T^n y) \neq 0$, then with (3.1) for $n \geq 2$, we have

$$\begin{aligned} d(T^n x, y) &= d(T^n x, T^n y) \\ &\leq \varepsilon d(T^{n-1}x, y) + \delta[d(T^{n-1}x, y) + d(y, T^n x)] + \gamma \frac{d(T^{n-1}x, T^n x)d(y,y)}{d(T^{n-1}x, y)} \\ &\quad + \beta \frac{d(T^{n-1}x, T^n x)d(y,y)}{d(T^{n-1}x, y) + d(T^{n-1}x, y) + d(y, T^n x)} + \alpha \frac{d(T^{n-1}x, T^n x)d(T^{n-1}x, y) + d(y, T^n x)d(y,y)}{d(T^n x, y) + d(y, T^{n-1}x)} \end{aligned}$$

By triangular inequality $d(T^n x, T^{n-1}x) \leq d(T^n x, y) + d(y, T^{n-1}x)$, we have

$d(T^n x, y) \leq \varepsilon d(T^{n-1}x, y) + \delta[d(T^{n-1}x, y) + d(y, T^n x)] + \alpha d(T^{n-1}x, y)$.

implies that

$$d(T^n x, y) \leq \left(\frac{\varepsilon + \delta + \alpha}{1 - \delta}\right) d(T^{n-1}x, y).$$

Then, by induction, we have

$$d(T^n x, y) \le \left(\frac{\varepsilon + \delta + \alpha}{1 - \delta}\right)^n d(x, y).$$

Using $\varepsilon + 2\delta + \gamma + \beta + \alpha < 1$ and taking limit as $n \to +\infty$ in the above inequality, we get

$$\lim_{n\to+\infty} T^n x = y.$$

Similarly, we get

$$\lim_{n\to+\infty} T^n x = z.$$

Now, the uniqueness of the limit gives that $y = z$.

(ii) If $d(T^{n-1}y, T^n x) + d(T^{n-1}x, T^n y) = 0$, then using (3.1), we have $d(T^n x, y) = 0$. Therefore,

$$\lim_{n\to+\infty} T^n x = y.$$

Similarly, we get

$$\lim_{n\to+\infty} T^n x = z.$$

Again, the uniqueness of the limit gives that $y = z$. Thus, T has a unique fixed point in X.

Example 3.5. Suppose $X = \{(0,0), (\frac{1}{2}, 0), (0,1)\}$ is a subset of $\mathbb{R}^2$ with the order $\le$ define as: for $(x_1, y_1), (x_2, y_2) \in X$ with $(x_1, y_1) \preccurlyeq (x_2, y_2)$ if and only if $x_1 \preccurlyeq x_2$ and $y_1 \preccurlyeq y_2$. Let the distance function $d: X \times X \to \mathbb{R}$ be defined by

$$d((x_1, y_1), (x_2, y_2)) = max\{|x_1 - x_2|, |y_1 - y_2|\}. \tag{3.7}$$

Again, let $d: X \to X$ be defined by $T(0,0) = (0,0)$, $T(0,1) = (\frac{1}{2}, 0)$ and $T(\frac{1}{2}, 0) = (0,0)$. Observe thar all the conditions of Theorem 3.1 and 3.3 are satisfied and (0,0) is the unique fixed point of T.

Corollary 3.6. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that $T: X \to X$ is a non-decreasing, continuous self mapping satisfying

$$d(Tx, Ty) \le \varepsilon d(x, y) + \delta[d(x, Ty) + d(y, Tx)] + \gamma \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} + \alpha \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \tag{3.8}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, and $\alpha, \beta, \gamma, \delta, \varepsilon \ge 0$ with $\varepsilon + 2\delta + \gamma + \beta + \alpha < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

Corollary 3.6 is still valid for T, not necessarily continuous, by assuming the following hypothesis in X.

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then

$$x = sup\{x_n\}. \tag{3.9}$$

Corollary 3.7. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that $T: X \to X$ is a monotone non-decreasing self mapping satisfying

$$\begin{aligned} d(Tx, Ty) \le \varepsilon d(x, y) + \delta[d(x, Ty) + d(y, Tx)] + \gamma \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ + \alpha \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{aligned} \tag{3.10}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, where $K = d(y, Tx) + d(x, Ty)$ and $\alpha, \beta, \gamma, \delta, \varepsilon \geq 0$ with $\varepsilon + 2\delta + \gamma + \beta + \alpha < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

Corollary 3.8. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that $T: X \to X$ is a non-decreasing self mapping and that for some positive integer m, self mapping T satisfying

$$\begin{aligned} d(T^m x, T^m y) \leq \varepsilon d(x, y) + \delta[d(x, T^m y) + d(y, T^m x)] + \gamma \frac{d(x, T^m x) d(y, T^m y)}{d(x, y)} \\ + \beta \frac{d(x, T^m x) d(y, T^m y)}{d(x, y) + d(x, T^m y) + d(y, T^m x)} + \alpha \frac{d(x, T^m x) d(x, T^m y) + d(y, T^m x) d(y, T^m y)}{d(y, T^m x) + d(x, T^m y)} \end{aligned} \tag{3.11}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, and $\alpha, \beta, \gamma, \delta, \varepsilon \geq 0$ with $\varepsilon + 2\delta + \gamma + \beta + \alpha < 1$. Suppose there exists $x_0 \in X$ with $x_0 \preccurlyeq T^m x_0$. If T^m is continuous, then T has a fixed point in X.

4. Application

In this section, we consider some of the application of our results for the self mapping involving integral type contractions areas follows:

Theorem 4.1. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that $T: X \to X$ is a non-decreasing, continuous self mapping satisfying

$$\begin{aligned} \int_0^{d(Tx,Ty)} ds \leq \varepsilon \int_0^{d(x,y)} ds + \delta \int_0^{[d(x,Ty)+d(y,Tx)]} ds + \gamma \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}} ds + \beta \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}} ds \\ + \alpha \int_0^{\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}} ds \end{aligned} \tag{4.1}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, and $\alpha, \beta, \gamma, \delta, \varepsilon \geq 0$ with $\varepsilon + 2\delta + \gamma + \beta + \alpha < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

Corollary 4.2. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that $T: X \to X$ is a non-decreasing, continuous self mapping satisfying

$$\int_0^{d(Tx,Ty)} ds \leq \varepsilon \int_0^{d(x,y)} ds + \delta \int_0^{[d(x,Ty)+d(y,Tx)]} ds + \gamma \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}} ds + \beta \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}} ds \tag{4.2}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, and $\beta, \gamma, \delta, \varepsilon \geq 0$ with $\varepsilon + 2\delta + \gamma + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

Corollary 4.3. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that $T: X \to X$ is a non-decreasing, continuous self mapping satisfying

$$\int_0^{d(Tx,Ty)} ds \leq \varepsilon \int_0^{d(x,y)} ds + \delta \int_0^{[d(x,Ty)+d(y,Tx)]} ds + \gamma \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}} ds + \alpha \int_0^{\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}} ds \tag{4.3}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, and $\alpha, \gamma, \delta, \varepsilon \geq 0$ with $\varepsilon + 2\delta + \gamma + \alpha < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

Corollary 4.4. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that $T: X \to X$ is a non-decreasing, continuous self mapping satisfying

$$\int_0^{d(Tx,Ty)} ds \le \varepsilon \int_0^{d(x,y)} ds + \delta \int_0^{[d(x,Ty)+d(y,Tx)]} ds + \beta \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}} ds + \alpha \int_0^{\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}} ds \tag{4.4}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, and $\alpha, \beta, \delta, \varepsilon \ge 0$ with $\varepsilon + 2\delta + \beta + \alpha < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

Corollary 4.5. Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that $T: X \to X$ is a non-decreasing, continuous self mapping satisfying

$$\int_0^{d(Tx,Ty)} ds \le \delta \int_0^{[d(x,Ty)+d(y,Tx)]} ds + \gamma \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}} ds + \beta \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}} ds + \alpha \int_0^{\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}} ds \tag{4.5}$$

for all distinct $x, y \in X$ with $y \preccurlyeq x$, and $\alpha, \beta, \gamma, \delta \ge 0$ with $2\delta + \gamma + \beta + \alpha < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

5. Remark: If $\gamma = 0, \beta = 0$ in Theorem 3.1, 3.3 and 3.4, then we obtain Theorem 31, 32 and 35 of [5].

6. Conclusion

In this paper, we considered rational type contraction mappings in metric spaces that are partially ordered. We demonstrated that such mappings have a unique fixed point, and we presented several results that support this conclusion. We also presented examples to highlight the improvements made over previous work on this topic.

Conflict of Interest

The authors declare that there are no conflict of interest.

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