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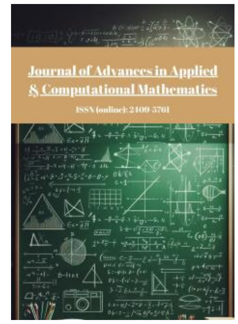

Sign-changing Solutions for Fourth Order Elliptic Equation with Concave-convex Nonlinearities

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ABSTRACT

In this paper, we study the following fourth order elliptic equation

$$\Delta^2 u - \Delta u + V(x)u = \kappa(x)|u|^{q-2}u + |u|^{p-2}u \ in \ R^N,$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $N > 4$, $1 < q < 2 < p < 2_* := \frac{2N}{N-4}$. Assuming that $V(x)$ satisfies a class of coercive conditions and the nonnegative weighted function $\kappa(x)$ belongs to $L^{\frac{p}{p-q}}(R^N)$, we obtain the existence of one sign-changing solution with the help of constraint variational method and quantitative deformation lemma. The novelty of this paper is that when the nonlinearity is the combination of concave and convex functions, we are able to obtain the existence of sign-changing solutions. Some recent results are improved and generalized significantly.

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1. Introduction

This paper is concerned with the following fourth order elliptic equation

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = \kappa(x)|u|^{q-2}u + |u|^{p-2}u \quad in\ R^N, \\ u \in H^2(R^N), \end{cases} \tag{1.1}$$

where $N > 4$, $1 < q < 2 < p < 2_* := \frac{2N}{N-4}$, the nonnegative potential function $\kappa(x)$ belongs to $L^{\frac{p}{p-q}}(R^N)$ and $V(x)$ satisfies the following coercive condition:

- $V(x) \in C(R^N, R)$ satisfies $\inf_{x\in R^N} V(x) \geq a > 0$, where a is a constant and $\lim_{|x|\to\infty} V(x) = +\infty$.

Fourth order elliptic equations are widely utilized in the description of various phenomena in physics, engineering and other sciences. When Ω is a smooth bounded domain in R^N, the problem

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) \quad & in\ \Omega, \\ u = \Delta u = 0 & on\ \partial\Omega \end{cases} \tag{1.2}$$

arises in many applications of mathematical physics and is often used to describe some phenomenon that occurs in various physics, engineering, and other sciences. In [1, 2], Lazer and Mckenna modelled suspension bridges using problem (1.2), which takes into account the fact that the coupling provided by the stays connecting the suspension cable to the deck of the road bed is fundamentally nonlinear. It was pointed out in [3-5] that problem (1.2) furnishes a good model for studying the static deflection of an elastic plate in a fluid. In addition, Ahmed and Harbi presented that problem (1.2) can also be applied to engineering, such as communication satellites, space shuttles and space stations equipped with large antennas mounted on long flexible beams in [6]. It is because of its deep background in mathematics and physics that many mathematicians have focused their attention on the existence or multiplicity of nontrivial solutions to fourth order elliptic problems on bounded domains, see [7-14] and the references listed therein. Compared with the bounded domains, when the whole space R^N is involved, the research for fourth order elliptic equations becomes more difficult, due to the lack of compactness of Sobolev embedding theorem. In spite of this, many authors were still committed to the study of this problem and searched for various hypotheses on $f(x,u)$ to ensure the existence of solutions for problem (1.2) or more general form. We refer the interested readers to [15-28] and the related works mentioned there.

However, it is worth pointing out that among the above papers only [29, 30] are concerned with the types of problems concerning the existence of sign-changing solutions of fourth order elliptic equations (1.1). In order to conveniently state our motivation, we present the detailed descriptions of these two papers. Explicitly, in [30] Pimenta discussed the existence of solutions for the following problem

$$\begin{cases} \Delta^2 u + V(x)u = f(x,u) \quad in\ R^N, \\ u \in H^2(R^N), \end{cases} \tag{1.3}$$

where $N \geq 5$, f and V satisfy the following assumption set:

- $0 < V_0 := \inf_{R^N} V$ and $V(x) = V(|x|)$ for all $x \in R^N$;
- $f: R^N \times R \to R$ is a Carathéodory function;
- $f(x,s) = o(|s|)$ as $s \to 0$ a.e. in R^N;
- there are constants $c_1, c_2 > 0$ and $0 < p < 2_* - 2$, where $2_* = \frac{2N}{N-4}$, such that

$$|f(x,s) - f(x,t)| \leq (c_1 + c_2(|s|^p + |t|^p))|s-t|, \ for\ a.e.\ x \in R^N\ and\ s,t \in R;$$

- $\lim\limits_{|s|\to\infty} \dfrac{F(x,s)}{s^2} = +\infty$ a.e. in R^N, where $F(x,s) = \int_0^s f(x,t)dt$;

- $\frac{f(x,s)}{s}$ is nondecreasing for $s > 0$ and nonincreasing for $s < 0$, for a.e. $x \in R^N$.

Under the above hypotheses, they obtained three radial solutions of problem (1.3): one being positive, one negative and one sign changing, see its Theorem 1.3. Che and Chen [29] dealt with the following fourth order elliptic equation

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = |u|^{p-1}u \quad in\ R^N, \\ \qquad\qquad u \in H^2(R^N), \end{cases} \tag{1.4}$$

where $p \in (2, 2_* - 1)$, $u: R^N \to R$, $V(x)$ is supposed to satisfy the following conditions:

- $V(x) \in C(R^N, R)$, $\inf_{x\in R^N} V(x) \geq a > 0$, where a is a constant;

- for each $M > 0$, $meas\,\{x \in R^N: V(x) \leq M\} < \infty$, where $meas(\cdot)$ denotes the Lebesgue measure in R^N.

Under the assumptions of (V_1) and (V_2), they showed the existence of least energy sign-changing solution to problem (1.4), see its Theorem 1.3. For some other types of fourth order elliptic equations in R^N, we refer the interested readers to [31-36] for more details.

In the light of the assumption (f_5) in [30] and $|u|^{p-1}$ with $2 < p < 2_* - 1$ in [29], the form $\kappa(x)|u|^{q-2}u + |u|^{p-2}u$ with $1 < q < 2 < p < 2_* := \frac{2N}{N-4}$, proposed in our problem (1.1), has not been investigated. Observing this fact, in the present paper, we focus our attention on this type of nonlinearity. At this position, it is necessary for us to mention the very recent work [18]. In fact, take $V(x) \equiv 1$ in equation (1.1) and suppose that

- $\kappa \in C(R^N) \cap L^{\frac{p}{p-q}}(R^N)$ with $\kappa^+ = max\{\kappa(x), 0\} \not\equiv 0$;

- $\int_{R^N} \kappa(x)|\tilde{u}|^q dx > 0$, where $\tilde{u}$ is the extremal function for the embedding of $H^2(R^N) \hookrightarrow L^p(R^N)$,

it was proved that equation (1.1) possesses one ground state solution when $0 < \|\kappa\|_{\frac{p}{p-q}} < \Lambda$ and admited at least two solutions for the case $0 < \|\kappa\|_{\frac{p}{p-q}} < \Lambda^*$, where

$$\Lambda := \left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_p^{-\frac{p(2-q)}{p-2} - q} > 0 \quad and \quad \Lambda^* := \frac{q}{2}\Lambda \tag{1.5}$$

with S_p defined in Lemma 2.1 below. However, the authors in [29] did not say any more about the sign of solutions obtained by themselves. As a result, inspired by the above papers, particularly [18, 29, 30], in this paper we investigate the existence of sign-changing solutions for problem (1.1) with concave-convex nonlinearities in R^N.

In the following, in order to conveniently state our main results, we need to recall some standard symbols. Define

$$H^2(R^N) := \{u \in L^2(R^N): \Delta u, \nabla u \in L^2(R^N)\}$$

and

$$E := \{u \in H^2(R^N): \int_{R^N} V(x)u^2 dx < \infty\}.$$

Then, under the assumption of (V_0), E is a Hilbert space with the following norm

$$\|u\|:=\left(\int_{R^N}(|\Delta u|^2+|\nabla u|^2+V(x)|u|^2)dx\right)^{\frac{1}{2}}.$$

Let $L^s(R^N)$ $(1\le s<\infty)$ be the Lebesgue space normed by

$$\|u\|_s:=\left(\int_{R^N}|u|^s dx.\right)^{\frac{1}{s}}.$$

To solve the problem (1.1) using variational methods, we introduce the corresponding energy functional

$$I(u):=\frac{1}{2}\|u\|^2-\frac{1}{q}\int_{R^N}\kappa(x)|u|^q dx-\frac{1}{p}\int_{R^N}|u|^p dx, \forall u\in E. \tag{1.6}$$

Under our assumptions on $V(x)$ and $\kappa(x)$, it is standard to verify that $I\in C^1(E,R)$ (see Lemma 2.3) and

$$\langle I'(u),\phi\rangle = \int_{R^N}(\Delta u\Delta\phi+\nabla u\nabla\phi+V(x)u\phi)dx-\int_{R^N}\kappa(x)|u|^{q-2}u\phi dx-\int_{R^N}|u|^{p-2}u\phi dx \tag{1.7}$$

for any u, $\phi\in E$.

We call that u is one (weak) solution of equation (1.1), if $u\in E$ is a critical point of I. In addition, if u is a solution of equation (1.1) with $u^{\pm}\neq 0$, then we say that u is a sign-changing solution, where $u^+(x):=max\{u(x),0\}$ and $u^-(x):=min\{u(x),0\}$.

Now, we are in the position to state our main result.

Theorem 1.1: Suppose that (V_0) holds and $0<\|\kappa\|_{\frac{p}{p-q}}<\Lambda^*$, then problem (1.1) admits one sign-changing solution.

Remark 1.2: In [18], the authors have studied the existence of positive solutions to the fourth order elliptic equation with concave-convex nonlinearities. However, they did not consider the existence of sign-changing solution to this type of nonlinearities. Therefore, in this paper, we use the method of [37] to deal with the sign-changing solutions of problem (1.1). To this point, we see that the technique is applicable for both local and non-local elliptic equations with such type nonlinear terms. As we observed, the authors in [18] not only discussed the existence of ground state solution, but also explored the existence of two nontrivial solutions, when the potential function $\kappa(x)$ is supposed to satisfy different requirements respectively, see its Theorems 1.1 and 1.2. Therefore, it is an interesting issue to search for additional assumptions on $\kappa(x)$ to guarantee the second nontrivial sign-changing solution.

Example 1.3: Let $N=5$, then $1<q<2<p<10=\frac{10}{5-4}$. Choose $q=\frac{3}{2}$, $p=4$, $\kappa(x)=\frac{\mu}{1+|x|^5}\in L^{\frac{8}{5}}(R^5)$ with $\mu>0$ small enough, it is easy to verify that the assumptions needed in our Theorem 1.1 are satisfied.

In [29], to obtain the existence of sign-changing solutions to problem (1.4), the authors investigated the minimizer corresponding to the following constraint minimization

$$m:=\inf_{u\in\mathcal{M}} I(u), \tag{1.8}$$

where $\mathcal{M}$ is the Nehari sign-changing manifold defined below

$$\mathcal{M}:=\{u\in E: u^{\pm}\neq 0, \langle I'(u),u^{\pm}\rangle=0\}.$$

During the arguments to ensure the achievement of the minimizer of problem (1.8), the projection property for every $u\in E$ with $u^{\pm}\neq 0$ on $\mathcal{M}$ plays an essential role. Explicitly, for every $u\in E$ with $u^{\pm}\neq 0$, there exits a unique couple of $(s_u, t_u) \in R^+ \times R^+$ such that $s_u u^+ + t_u u^- \in \mathcal{M}$. However, for our problem (1.1), due to the appearance of the convex term $\kappa(x)|u|^{q-2}$ with $1 < q < 2$, the above projection property of $u^\pm \neq 0$ on $\mathcal{M}$ is no longer valid. In other words, it is not feasible to select $\mathcal{M}$ as the constraint set to find the existence of sign-changing solutions for our problem (1.1). In order to overcome this difficulty, we construct one new non-empty closed subset $\widetilde{\mathcal{M}}^- \subset \mathcal{M}$ and prove the related properties of $\widetilde{\mathcal{M}}^-$, for example, we can find a pair of unique maximums on $\widetilde{\mathcal{M}}^-$ (see Lemma 3.3). Then, we are allowed to consider the corresponding infimum problem

$$\widetilde{m} := \inf_{u \in \widetilde{\mathcal{M}}^-} I(u) \tag{1.9}$$

Thereafter, by means of the quantitative deformation lemma and the Brower degree theory, we are allowed to show that the minimizer of infimum problem (1.9) is just one sign-changing solution of problem (1.1).

We use "→" and "⇀" to denote the strong and weak convergence in the related function spaces, respectively. Let $R^+ = [0, \infty)$ be the set of nonnegative real numbers. $\langle \cdot, \cdot \rangle$ denote the dual pair for any Banach space and its dual space. $X \hookrightarrow Y$ means X embeds into Y.

The rest of the study is organized as follows. In Section 2, we present preliminary results and demonstrate the non-emptiness of the constraint set $\widetilde{\mathcal{M}}^-$ defined in (2.10) below. In Section 3, we are concerned with the proof of Theorem 1.1.

2. Preliminaries

Now, we recall the following continuous and compact embedding conclusions.

Lemma 2.1: ([38, Lemma 2.1]) Under the assumption of (V_0), the embedding $E \hookrightarrow L^r(R^N)$ is compact for $2 \leq r < 2_*$ and $E \hookrightarrow L^r(R^N)$ is continuous for $2 \leq r \leq 2_*$, that is, there exists $S_r > 0$ depend on r such that

$$\|u\|_r \leq S_r \|u\|, \quad \forall u \in E. \tag{2.1}$$

The next lemma is related to the convergence of the convex terms $\kappa(x)|u|^q$.

Lemma 2.2: If $u_n \rightharpoonup u$ in E, then, up to a subsequence, we have

$$\int_{R^N} \kappa(x)|u_n|^q dx \to \int_{R^N} \kappa(x)|u|^q dx \tag{2.2}$$

Proof. If $u_n \rightharpoonup u$ in E, we deduce that $\{u_n\}$ is bounded uniformly in E. Therefore, by Lemma 2.1, up to a subsequence, $\{u_n\}$ converges strongly to u in $L^p(R^N)$ for $2 < p < 2_*$. Notice that $\kappa(x)$ is nonnegative, so we use the triangle inequality of L^q norm and Hölder inequality with $\frac{1}{q} = \frac{p-q}{pq} + \frac{1}{p}$ to deduce that

$$\begin{aligned}
& \left| \left(\int_{R^N} \kappa(x)|u_n(x)|^q dx \right)^{\frac{1}{q}} - \left(\int_{R^N} \kappa(x)|u(x)|^q dx \right)^{\frac{1}{q}} \right| \\
= & \left| \left(\int_{R^N} |\kappa(x)^{\frac{1}{q}} u_n(x)|^q dx \right)^{\frac{1}{q}} - \left(\int_{R^N} |\kappa(x)^{\frac{1}{q}} u(x)|^q dx \right)^{\frac{1}{q}} \right| \\
\leq & \left| \left(\int_{R^N} |\kappa(x)^{\frac{1}{q}} u_n(x) - \kappa(x)^{\frac{1}{q}} u(x)|^q dx \right)^{\frac{1}{q}} \right| \\
\leq & \left(\int_{R^N} \kappa(x)^{\frac{p}{p-q}} dx \right)^{\frac{p-q}{pq}} \left(\int_{R^N} |u_n(x) - u(x)|^p dx \right)^{\frac{1}{p}},
\end{aligned}$$

which implies (2.2).

Lemma 2.3: The energy functional $I \in C^1(E, R)$.

Proof. In terms of (1.6), let $I_1 = \|u\|^2$, $I_2 = \int_{R^N} |u|^p dx$ and $I_3 = \int_{R^N} \kappa(x)|u|^q dx$. It is obvious that $I_1 \in C^1(E, R)$. According to the proof of ([39, Proposition 1.12]), one can get $I_2 \in C^1(E, R)$. In the following, we only need to verify $I_3 \in C^1(E, R)$.

Existence of the Gateaux Derivative: Let u, $v \in E$. Given $x \in R^N$ and $0 < |t| < 1$, by the mean value theorem, there exists $\lambda \in [0,1]$ such that

$$\begin{aligned}\frac{|\kappa(x)(|u(x)+tv(x)|^q - |u(x)|^q)|}{|t|} &= q|\kappa(x)||u(x)+\lambda t v(x)|^{q-1}|v(x)| \\ &\leq q|\kappa(x)|[|u(x)|+|v(x)|]^{q-1}|v(x)|.\end{aligned}$$

By the Hölder inequality and (2.1), we get that

$$\begin{aligned}|\kappa(x)||u(x)+v(x)|^{q-1}|v(x)| &\leq ||\kappa||_{\frac{p}{p-q}} |||u|+|v|||_p^{q-1} ||v||_p \\ &\leq ||\kappa||_{\frac{p}{p-q}} (||u||_p^{q-1} + ||v||_p^{q-1}) ||v||_p \\ &\leq ||\kappa||_{\frac{p}{p-q}} (S_p^{q-1}||u||^{q-1} + S_p^{q-1}||v||^{q-1}) S_p ||v||,\end{aligned}$$

which means that $|\kappa(x)||u(x)+v(x)|^{q-1}|v(x)| \in L^1(R^N)$. It follows from that the Lebesgue theorem that

$$\langle I_3{}'(u), v\rangle = \int_{R^N} q\kappa(x)|u|^{q-2}uv dx.$$

Continuity of the Gateaux Derivative. Assume that $u_n \to u$ in E, it is equivalent to show that $\langle I_3{}'(u_n), v\rangle \to \langle I_3{}'(u), v\rangle$ as $n \to \infty$ for any given $v \in E$. Suppose by contradiction that there exists one subsequence, still denoted by $\{u_n\}$, such that $\langle I_3{}'(u_n), v\rangle \nrightarrow \langle I_3{}'(u), v\rangle$. From the continuity of embedding $E \hookrightarrow L^s(R^N)$ for every $s \in [2, 2_*]$, the sequence $\{|u_n|^{q-2}u_n\}$ is bounded in $L^{\frac{p}{q-1}}(R^N)$. Moreover, up to a subsequence if necessary, assume that $u_n(x) \to u(x)$ almost everywhere in R^N. Then, we see that

$$|u_n|^{q-2}u_n \rightharpoonup |u|^{q-2}u \ in \ L^{\frac{p}{q-1}}(R^N).$$

Meanwhile, from the Hölder inequality and $\kappa \in L^{\frac{p}{p-q}}(R^N)$, it is obvious to infer that $\kappa v \in L^{\frac{p}{p-(q-1)}}(R^N)$. As a consequence, there holds that

$$\int_{R^N} q\kappa(x)|u_n|^{q-2}u_n v dx \to \int_{R^N} q\kappa(x)|u|^{q-2}uv dx,$$

which is an evident contradiction.

In the subsequent of this section, we introduce the effective set $\widetilde{\mathcal{M}}^-$ for problem (1.1). To this end, for fixed $u \in E\backslash\{0\}$, we need to consider the fibering map $\psi_u: [0, \infty) \to R$ given by

$$\psi_u(r) := \frac{r^2}{2}||u||^2 - \frac{r^q}{q}\int_{R^N} \kappa(x)|u|^q dx - \frac{r^p}{p}\int_{R^N} |u|^p dx = I(ru), \forall r \geq 0.$$

Then, direct calculations give that

$$\psi_u^{'}(r) = r||u||^2 - r^{q-1}\int_{R^N} \kappa(x)|u|^q dx - r^{p-1}\int_{R^N} |u|^p dx = \langle I^{'}(ru), u\rangle \tag{2.3}$$

and

$$\psi_u''(r) = ||u||^2 - (q-1)r^{q-2}\int_{R^N}\kappa(x)|u|^q dx - (p-1)r^{p-2}\int_{R^N}|u|^p dx \tag{2.4}$$

Obviously, one has

$$\psi_u''(1) = ||u||^2 - (q-1)\int_{R^N}\kappa(x)|u|^q dx - (p-1)\int_{R^N}|u|^p dx \tag{2.5}$$

Meanwhile, in view of (2.3), we see that the Nehari manifold $\mathcal{N}$ can be determined equivalently by $\psi_u'(1) = 0$, that is,

$$\mathcal{N} = \{u \in E\backslash\{0\}: \langle I'(u), u\rangle = 0\} = \{u \in E\backslash\{0\}: \psi_u'(1) = 0\}. \tag{2.6}$$

In addition, it is necessary for us to consider the function $g_u: [0,\infty)\times[0,\infty) \to R$ defined as $g_u(s,t) := I(su^+ + tu^-)$ for any $u \in E$ with $u^\pm \neq 0$. From (1.6), $g_u(s,t)$ is of the following form

$$\begin{aligned} g_u(s,t) \quad &= \frac{s^2}{2}||u^+||^2 - \frac{s^q}{q}\int_{R^N}\kappa(x)|u^+|^q dx - \frac{s^p}{p}\int_{R^N}|u^+|^p dx \\ &+ \frac{t^2}{2}||u^-||^2 - \frac{t^q}{q}\int_{R^N}\kappa(x)|u^-|^q dx - \frac{t^p}{p}\int_{R^N}|u^-|^p dx \\ &=: I(su^+) + I(tu^-). \end{aligned} \tag{2.7}$$

It is easy to see that

$$\frac{\partial g_u(s,t)}{\partial s} = s||u^+||^2 - s^{q-1}\int_{R^N}\kappa(x)|u^+|^q dx - s^{p-1}\int_{R^N}|u^+|^p dx, \tag{2.8}$$

$$\frac{\partial g_u(s,t)}{\partial t} = t||u^-||^2 - t^{q-1}\int_{R^N}\kappa(x)|u^-|^q dx - t^{p-1}\int_{R^N}|u^-|^p dx, \tag{2.9}$$

$$\frac{\partial^2 g_u(s,t)}{\partial s^2} = \quad ||u^+||^2 - (q-1)s^{q-2}\int_{R^N}\kappa(x)|u^+|^q dx - (p-1)s^{p-2}\int_{R^N}|u^+|^p dx,$$

$$\frac{\partial^2 g_u(s,t)}{\partial t^2} = \quad ||u^-||^2 - (q-1)t^{q-2}\int_{R^N}\kappa(x)|u^-|^q dx - (p-1)t^{p-2}\int_{R^N}|u^-|^p dx$$

and

$$\frac{\partial^2 g_u(s,t)}{\partial s\partial t} = 0 = \frac{\partial^2 g_u(s,t)}{\partial t\partial s}.$$

Based on the above notations, the new constraint set $\widetilde{\mathcal{M}}^-$ is defined as below:

$$\widetilde{\mathcal{M}}^- := \left\{u \in \mathcal{M}: \frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(1,1)} < 0, \frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(1,1)} < 0\right\}. \tag{2.10}$$

Remark 2.4: Note that

$$\begin{aligned} \psi_u''(1) \quad &= ||u||^2 - (q-1)\int_{R^N}\kappa(x)|u|^q dx - (p-1)\int_{R^N}|u|^p dx \\ &= \frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(1,1)} + \frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(1,1)}. \end{aligned}$$

Therefore, we get

$$\psi_u''(1) < 0, \forall u \in \widetilde{\mathcal{M}}^- \tag{2.11}$$

For the convenience of the consequent discussions, we need to tell more about $g_u(s,t)$. To begin with, from the definition of $g_u(s,t)$, we observe that

$$\frac{\partial g_u(s,t)}{\partial s} = \langle I^{'}(su^+ + tu^-), u^+\rangle \quad and \quad \frac{\partial g_u(s,t)}{\partial t} = \langle I^{'}(su^+ + tu^-), u^-\rangle.$$

Thus, for any $u \in E$ with $u^{\pm} \neq 0$, $su^+ + tu^- \in \mathcal{M}$ if and only if $\frac{\partial g_u(s,t)}{\partial s} = 0$ and $\frac{\partial g_u(s,t)}{\partial t} = 0$, that is,

$$\mathcal{M} = \left\{u \in E: u^{\pm} \neq 0, \frac{\partial g_u(s,t)}{\partial s}|_{(1,1)} = 0, \frac{\partial g_u(s,t)}{\partial t}|_{(1,1)} = 0\right\}. \tag{2.12}$$

Furthermore, for any $u \in \mathcal{M}$, we infer that

$$\begin{aligned} \frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(1,1)} = &\quad ||u^+||^2 - (q-1)\int_{R^N} \kappa(x)|u^+|^q dx - (p-1)\int_{R^N}|u^+|^p dx \\ = &\quad (2-p)||u^+||^2 - (q-p)\int_{R^N}\kappa(x)|u^+|^q dx \\ = &\quad (2-q)||u^+||^2 - (p-q)\int_{R^N}|u^+|^p dx, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(1,1)} = &\quad ||u^-||^2 - (q-1)\int_{R^N} \kappa(x)|u^-|^q dx - (p-1)\int_{R^N}|u^-|^p dx \\ = &\quad (2-p)||u^-||^2 - (q-p)\int_{R^N}\kappa(x)|u^-|^q dx \\ = &\quad (2-q)||u^-||^2 - (p-q)\int_{R^N}|u^-|^p dx \end{aligned} \tag{2.14}$$

and

$$\frac{\partial^2 g_u(s,t)}{\partial s \partial t}|_{(1,1)} = \frac{\partial^2 g_u(s,t)}{\partial t \partial s}|_{(1,1)} = 0. \tag{2.15}$$

The following lemma shows that $\widetilde{\mathcal{M}}^-$ is nonempty.

Lemma 2.5: If $0 < \|\kappa\|_{\frac{p}{p-q}} < \Lambda$ and $u \in E$ with $u^{\pm} \neq 0$, there exists one pair $(s_{u^+}, t_{u^-}) \in (0,\infty) \times (0,\infty)$ such that $s_{u^+}u^+ + t_{u^-}u^- \in \widetilde{\mathcal{M}}^-$, which is one local maximum point of $g_u(s,t)$.

Proof. Fixing $u \in E$ with $u^{\pm} \neq 0$, we note that

$$\begin{aligned} \frac{\partial g_u(s,t)}{\partial s} &= s||u^+||^2 - s^{p-1}\int_{R^N}|u^+|^p dx - s^{q-1}\int_{R^N}\kappa(x)|u^+|^q dx \\ &= s^{q-1}(s^{2-q}||u^+||^2 - s^{p-q}\int_{R^N}|u^+|^p dx - \int_{R^N}\kappa(x)|u^+|^q dx) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \frac{\partial g_u(s,t)}{\partial t} &= t||u^-||^2 - t^{p-1}\int_{R^N}|u^-|^p dx - t^{q-1}\int_{R^N}\kappa(x)|u^-|^q dx \\ &= t^{q-1}(t^{2-q}||u^-||^2 - t^{p-q}\int_{R^N}|u^-|^p dx - \int_{R^N}\kappa(x)|u^-|^q dx). \end{aligned} \tag{2.17}$$

Define $\zeta \in C([0,\infty), R)$ and $\gamma \in C([0,\infty), R)$ by

$$\zeta(s) := s^{2-q}||u^+||^2 - s^{p-q}\int_{R^N}|u^+|^p dx$$

and

$$\gamma(t) := t^{2-q}||u^-||^2 - t^{p-q}\int_{R^N}|u^-|^p dx.$$

By the straight computations, one has

$$\begin{aligned}\zeta'(s) &= (2-q)s^{1-q}||u^+||^2 - (p-q)s^{p-q-1}\int_{R^N}|u^+|^p dx\\ &= s^{1-q}((2-q)||u^+||^2 - (p-q)s^{p-2}\int_{R^N}|u^+|^p dx)\end{aligned}$$

and

$$\begin{aligned}\gamma'(t) &= (2-q)t^{1-q}||u^-||^2 - (p-q)t^{p-q-1}\int_{R^N}|u^-|^p dx\\ &= t^{1-q}((2-q)||u^-||^2 - (p-q)t^{p-2}\int_{R^N}|u^-|^p dx).\end{aligned}$$

Obviously, $\zeta(s)$ has a unique critical point

$$s_{max} := s_{max,u^+} = \left(\frac{(2-q)||u^+||^2}{(p-q)\int_{R^N}|u^+|^p dx}\right)^{\frac{1}{p-2}} > 0$$

and $\gamma(t)$ possesses a unique critical point

$$t_{max} := t_{max,u^-} = \left(\frac{(2-q)||u^-||^2}{(p-q)\int_{R^N}|u^-|^p dx}\right)^{\frac{1}{p-2}} > 0.$$

Moreover, $\zeta(s)$ is strictly increasing in (0, s_{max,u^+}) and strictly decreasing in (s_{max,u^+}, ∞), $\gamma(t)$ is strictly increasing in (0, t_{max,u^-}) and strictly decreasing in (t_{max,u^-}, ∞). That is, s_{max,u^+} is the maximum point of $\zeta(s)$ and t_{max,u^-} is the maximum point of $\gamma(t)$.

Therefore, under the assumption of $0 < ||\kappa||_{\frac{p}{p-q}} < \Lambda$, it follows from (2.1) that

$$\begin{aligned}\zeta(s_{max}) &= \left(\frac{(2-q)||u^+||^2}{(p-q)\int_{R^N}|u^+|^p dx}\right)^{\frac{2-q}{p-2}}||u^+||^2 - \left(\frac{(2-q)||u^+||^2}{(p-q)\int_{R^N}|u^+|^p dx}\right)^{\frac{p-q}{p-2}}\int_{R^N}|u^+|^p dx\\ &= \left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}}\left[\left(\frac{||u^+||^2}{\int_{R^N}|u^+|^p dx}\right)^{\frac{2-q}{p-2}}||u^+||^2 - \frac{2-q}{p-q}\left(\frac{||u^+||^2}{\int_{R^N}|u^+|^p dx}\right)^{\frac{p-q}{p-2}}\int_{R^N}|u^+|^p dx\right]\\ &= \frac{p-2}{p-q}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}}\frac{||u^+||^{\frac{2(p-q)}{p-2}}}{(\int_{R^N}|u^+|^p dx)^{\frac{2-q}{p-2}}}\\ &= ||u^+||^q\left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}}\left(\frac{||u^+||^p}{\int_{R^N}|u^+|^p dx}\right)^{\frac{2-q}{p-2}}\\ &\geq ||u^+||^q\left(\frac{p-2}{p-q}\right)\left(\frac{(2-q)S_p^{-p}}{p-q}\right)^{\frac{2-q}{p-2}}\\ &> ||\kappa||_{\frac{p}{p-q}}S_p^q||u^+||^q\\ &\geq \int_{R^N}\kappa(x)|u^+|^q dx,\end{aligned}$$

that is,

$$\zeta(s_{max}) > \int_{R^N} \kappa(x)|u^+|^q dx.$$

Similarly, we know that

$$\gamma(t_{max}) > \int_{R^N} \kappa(x)|u^-|^q dx.$$

As a consequence, for any $u \in E$ with $u^{\pm} \neq 0$, there exists exactly $0 < s_{1,u^+} < s_{max} < s_{2,u^+}$ such that

$$\zeta(s_{1,u^+}) - \int_{R^N} \kappa(x)|u^+|^q dx = 0, \quad \zeta(s_{2,u^+}) - \int_{R^N} \kappa(x)|u^+|^q dx = 0 \tag{2.18}$$

and

$$\zeta^{'}(s_{1,u^+}) > 0, \quad \zeta^{'}(s_{2,u^+}) < 0. \tag{2.19}$$

Meanwhile, there exists exactly $0 < t_{1,u^-} < t_{max} < t_{2,u^-}$ such that

$$\gamma(t_{1,u^-}) - \int_{R^N} \kappa(x)|u^-|^q dx = 0, \quad \gamma(t_{2,u^-}) - \int_{R^N} \kappa(x)|u^-|^q dx = 0 \tag{2.20}$$

and

$$\gamma^{'}(t_{1,u^-}) > 0, \quad \gamma^{'}(t_{2,u^-}) < 0. \tag{2.21}$$

Thus, it follows from (2.18) and (2.20) that (s_{1,u^+}, t_{1,u^-}), (s_{1,u^+}, t_{2,u^-}), (s_{2,u^+}, t_{1,u^-}) and (s_{2,u^+}, t_{2,u^-}) are the stationary points of $g_u(s,t)$.

In addition, we observe that

$$\frac{\partial^2 g_u(s,t)}{\partial s^2} = s^{q-1}\zeta^{'}(s) \quad and \quad \frac{\partial^2 g_u(s,t)}{\partial t^2} = t^{q-1}\gamma^{'}(t), \tag{2.22}$$

when $\frac{\partial g_u(s,t)}{\partial s}(s,t) = 0$ and $\frac{\partial g_u(s,t)}{\partial t}(s,t) = 0$. Therefore, (2.19), (2.21) and (2.22) indicate that

$$\frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(s_{1,u^+},t_{1,u^-})} > 0 \quad and \quad \frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(s_{1,u^+},t_{1,u^-})} > 0, \tag{2.23}$$

$$\frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(s_{1,u^+},t_{2,u^-})} > 0 \quad and \quad \frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(s_{1,u^+},t_{2,u^-})} < 0, \tag{2.24}$$

$$\frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(s_{2,u^+},t_{1,u^-})} < 0 \quad and \quad \frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(s_{2,u^+},t_{1,u^-})} > 0, \tag{2.25}$$

$$\frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(s_{2,u^+},t_{2,u^-})} < 0 \quad and \quad \frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(s_{2,u^+},t_{2,u^-})} < 0. \tag{2.26}$$

Moreover, in view of the fact $\frac{\partial^2 g_u(s,t)}{\partial s \partial t} = 0$, we are allowed to infer that (s_{1,u^+}, t_{1,u^-}) is one local minimum point of $g_u(s,t)$ and (s_{2,u^+}, t_{2,u^-}) is one local maximum point of $g_u(s,t)$, which is exactly the desired $(s_{u^+}, t_{u^-}) \in (0,\infty) \times (0,\infty)$.

Corollary 2.6: If $0 < ||\kappa||_{\frac{p}{p-q}} < \Lambda$, then for any $u \in \mathcal{M}$,

$$\frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(1,1)} \neq 0 \quad and \quad \frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(1,1)} \neq 0.$$

Proof. According to the proof of Lemma 2.5, we know that $g_u(s,t)$ has exactly four stationary points: (s_{1,u^+}, t_{1,u^-}), (s_{1,u^+}, t_{2,u^-}), (s_{2,u^+}, t_{1,u^-}) and (s_{2,u^+}, t_{2,u^-}). Meanwhile, since $u \in \mathcal{M}$, it follows from (2.12) that $\frac{\partial g_u(s,t)}{\partial s}|_{(1,1)} = 0$, which implies that $(s_{i,u^+}, t_{j,u^-}) = (1,1)$, $i, j = 1,2$. Therefore, (2.23)-(2.26) states that $\frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(1,1)} \neq 0$ and $\frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(1,1)} \neq 0$.

3. Proof of Theorem 1.1

Before entering the proof of Theorem 1.1, the following lemmas are presented in order. During this section, we assume that $0 < ||\kappa||_{\frac{p}{p-q}} < \Lambda^*$.

Lemma 3.1: If $0 < ||\kappa||_{\frac{p}{p-q}} < \Lambda^*$, then $\widetilde{m} > 0$.

Proof. Letting $u \in \widetilde{\mathcal{M}}^-$, in view of (2.11), we have

$$\begin{aligned} 0 &> \psi_u''(1) \\ &= \psi_u''(1) - (p-1)\langle I'(u), u\rangle \\ &= (2-q)||u||^2 - (p-q)\int_{R^N} |u|^p dx. \end{aligned}$$

Thus, according to (2.1), there holds that

$$\frac{2-q}{p-q}||u||^2 < \int_{R^N} |u|^p dx \leq S_p^p ||u||^p.$$

Subsequently, using the Hölder inequality and in view of $0 < ||\kappa||_{\frac{p}{p-q}} < \Lambda^*$, we deduce that

$$\begin{aligned} I(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)||u||^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int_{R^N} \kappa(x)|u|^q dx \\ &\geq \frac{p-2}{2p}||u||^2 - \frac{p-q}{pq} S_p^q ||\kappa||_{\frac{p}{p-q}} ||u||^q \\ &= ||u||^q \left(\frac{p-2}{2p}||u||^{2-q} - \frac{p-q}{pq} S_p^q ||\kappa||_{\frac{p}{p-q}}\right) \\ &> \left(\frac{2-q}{p-q}\right)^{\frac{q}{p-2}} S_p^{-\frac{pq}{p-2}} \left(\frac{p-2}{2p}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_p^{-\frac{p(2-q)}{p-2}} - \frac{p-q}{pq} S_p^q ||\kappa||_{\frac{p}{p-q}}\right) \\ &> \left(\frac{2-q}{p-q}\right)^{\frac{q}{p-2}} S_p^{-\frac{pq}{p-2}} \Bigg(\frac{p-2}{2p}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_p^{-\frac{p(2-q)}{p-2}} \\ &\quad - \left(\frac{p-q}{pq}\right) S_p^q \frac{q}{2}\left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_p^{-\frac{p(2-q)}{p-2}-q}\Bigg) \\ &= 0. \end{aligned}$$

That is, $\widetilde{m} > 0$.

Lemma 3.2: Assume that $0 < ||\kappa||_{\frac{p}{p-q}} < \Lambda^*$, for any $u \in \widetilde{\mathcal{M}}^-$, $\|u^\pm\| > (\frac{(2-q)S_p^{-p}}{p-q})^{\frac{1}{p-2}} > 0$.

Proof. For any $u \in \widetilde{\mathcal{M}}^-$, from $\frac{\partial^2 g_u(s,t)}{\partial s^2}|_{(1,1)} < 0$, $\frac{\partial^2 g_u(s,t)}{\partial t^2}|_{(1,1)} < 0$, (2.13) and (2.14), we derive

$$(2-q)||u^\pm||^2 < (p-q)\int_{R^N} |u^\pm|^p dx \leq (p-q) S_p^p ||u^\pm||^p,$$

which yields that

$$||u^{\pm}|| > \left(\frac{(2-q)S_p^{-p}}{p-q}\right)^{\frac{1}{p-2}} > 0.$$

Lemma 3.3: Assume that $0 < ||\kappa||_{\frac{p}{p-q}} < \Lambda^*$. If $u \in E$ with $u^{\pm} \neq 0$, there exists a unique pair $(s_u, t_u) \in (0,\infty) \times (0,\infty)$ such that $s_u u^+ + t_u u^- \in \widetilde{\mathcal{M}}^-$ and

$$I(s_u u^+ + t_u u^-) = \max_{s,t\geq 0} I(su^+ + tu^-).$$

Proof. On the one hand, in view of Lemma 2.5, for any $u \in H^2(R^N)$ with $u^{\pm} \neq 0$, we deduce that $g_u(s,t) = I(su^+ + tu^-)$ possesses exactly two pairs of extreme points (s_{1,u^+}, t_{1,u^-}) and (s_{2,u^+}, t_{2,u^-}), where (s_{1,u^+}, t_{1,u^-}) is one local minimum point and (s_{2,u^+}, t_{2,u^-}) is one local maximum point. On the other hand, Lemma 3.1 guarantees that

$$\max_{s,t\geq 0} I(su^+ + tu^-) > 0.$$

Additionally, from the expression of $g_u(s,t)$, that is, (2.7), it is easy to know that

$$g_u(s,t) \to 0 \;\; as \; |(s,t)| \to 0 \quad and \quad g_u(s,t) \to -\infty \;\; as \; |(s,t)| \to \infty.$$

From the above facts, we conclude that the maximum of $I(su^+ + tu^-)$ must be achieved at the local maximum point (s_{2,u^+}, t_{2,u^-}). Let $(s_u, t_u) := (s_{2,u^+}, t_{2,u^-})$, this proof is complete.

Lemma 3.4: Assume that $0 < ||\kappa||_{\frac{p}{p-q}} < \Lambda^*$, then $\widetilde{\mathcal{M}}^-$ is a closed set.

Proof. Let $\{u_n\} \subset \widetilde{\mathcal{M}}^-$ satisfy $u_n \to u_0$ as $n \to \infty$ in E. It is enough to show that $u_0 \in \widetilde{\mathcal{M}}^-$. It follows from $u_n \in \widetilde{\mathcal{M}}^-$ that

$$\langle I'(u_0), u_0^{\pm}\rangle = \lim_{n\to\infty} \langle I'(u_n), u_n^{\pm}\rangle = 0, \tag{3.1}$$

$$\frac{\partial^2 g_{u_0}(s,t)}{\partial s^2}\Big|_{(1,1)} = \lim_{n\to\infty} \frac{\partial^2 g_{u_n}(s,t)}{\partial s^2}\Big|_{(1,1)} \leq 0, \tag{3.2}$$

$$\frac{\partial^2 g_{u_0}(s,t)}{\partial t^2}\Big|_{(1,1)} = \lim_{n\to\infty} \frac{\partial^2 g_{u_n}(s,t)}{\partial t^2}\Big|_{(1,1)} \leq 0. \tag{3.3}$$

By Lemma 3.2, we deduce that

$$\|u_0^{\pm}\| = \lim_{n\to\infty} \|u_n^{\pm}\| \geq \left(\frac{(2-q)S_p^{-p}}{p-q}\right)^{\frac{1}{p-2}} > 0,$$

which implies that $u_0^{\pm} \neq 0$. Hence, from this and (3.1), one has $u_0 \in \mathcal{M}$. In addition, it follows from (3.2), (3.3) and Corollary 2.6 that

$$\frac{\partial^2 g_{u_0}(s,t)}{\partial s^2}\Big|_{(1,1)} < 0, \quad \frac{\partial^2 g_{u_0}(s,t)}{\partial t^2}\Big|_{(1,1)} < 0.$$

Hence, $u_0 \in \widetilde{\mathcal{M}}^-$, namely, $\widetilde{\mathcal{M}}^-$ is a closed set.

Lemma 3.5: If $0 < ||\kappa||_{\frac{p}{p-q}} < \Lambda^*$, the infimum $\widetilde{m}$ defined in (1.9) can be achieved by some $\bar{u} \in \widetilde{\mathcal{M}}^-$.

Proof. Let $\{u_n\} \subset \widetilde{\mathcal{M}}^-$ be a minimizing sequence for the functional I, namely, $I(u_n) \to \widetilde{m}$ as $n \to \infty$. Observing that

$$\begin{aligned} I(u_n) &= I(u_n) - \frac{1}{p}\langle I^{'}(u_n), u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)||u_n||^2 - \left(\frac{1}{q}-\frac{1}{p}\right)\int_{R^N}\kappa(x)|u_n|^q dx \\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)||u_n||^2 - S_p^q||\kappa||_{\frac{p}{p-q}}\left(\frac{1}{q}-\frac{1}{p}\right)||u_n||^q \end{aligned}$$

and $1 < q < 2 < p < 2_*$, we know that $\{u_n\}$ is bounded in E. Therefore, up to a subsequence, there exists $\bar{u} \in E$ such that $u_n \rightharpoonup \bar{u}$ in E and $u_n \to \bar{u}$ in $L^p(R^N)$.

First of all, we claim that $\bar{u}^{\pm} \neq 0$. From Lemmas 2.1, 2.2 and 3.2, we deduce that

$$\begin{aligned} \int_{R^N}\kappa(x)|\bar{u}^{\pm}|^q\, dx + \int_{R^N}|\bar{u}^{\pm}|^p\, dx &= \lim_{n\to\infty}(\int_{R^N}\kappa(x)|u_n^{\pm}|^q\, dx + \int_{R^N}|u_n^{\pm}|^p\, dx) \\ &= \lim_{n\to\infty}||u_n^{\pm}||^2 \\ &\geq \left(\frac{(2-q)S_p^{-p}}{p-q}\right)^{\frac{2}{p-2}} > 0, \end{aligned}$$

which yields that $\bar{u}^{\pm} \neq 0$. Hence, it remains to prove $u_n \to \bar{u}$ in E. In fact, if not, then we are allowed to assume that

$$\|\bar{u}^+\| < \liminf_{n\to\infty}\|u_n^+\| \quad or \quad \|\bar{u}^-\| < \liminf_{n\to\infty}\|u_n^-\|$$

Note that Lemma 3.3 states the existence of a unique pair of $(s_{\bar{u}}, t_{\bar{u}})$ such that

$$s_{\bar{u}}\bar{u}^+ + t_{\bar{u}}\bar{u}^- \in \widetilde{\mathcal{M}}^-,\ I(u_n^+ + u_n^-) = \max_{s,t\geq 0} I(su_n^+ + tu_n^-).$$

Therefore, we obtain that

$$\widetilde{m} \leq I(s_{\bar{u}}\bar{u}^+ + t_{\bar{u}}\bar{u}^-) < \liminf_{n\to\infty} I(s_{\bar{u}}u_n^+ + t_{\bar{u}}u_n^-) \leq \liminf_{n\to\infty} I(u_n^+ + u_n^-) = \widetilde{m}.$$

This obvious contradiction means that that $u_n \to \bar{u}$ in E. Due to the fact that $\widetilde{\mathcal{M}}^-$ is closed, one has $\bar{u} \in \widetilde{\mathcal{M}}^-$ and $\widetilde{m} = I(\bar{u})$.

Based on the above lemmas, we are in the position to obtain the sign-changing solution on $\widetilde{\mathcal{M}}^-$ via using the deformation lemma (see [39, Theorem 2.3]).

Proof of Theorem 1.1: Suppose that $\bar{u} \in \widetilde{\mathcal{M}}^-$ is the minimizer of the functional I obtained in Lemma 3.5. Next, it is enough to demonstrate that $\bar{u}$ is a critical point of the functional I, that is, $I^{'}(\bar{u}) = 0$. First of all, it follows from Lemma 3.3 that

$$I(\rho\bar{u}^+ + \tau\bar{u}^-) < I(\bar{u}^+ + \bar{u}^-) = \widetilde{m} = \inf_{u\in\widetilde{\mathcal{M}}^-} I(u),\ \forall(\rho,\tau) \in \mathrm{R}_+^2 \setminus\{(1,1)\}.$$

In addition, $\bar{u} \in \widetilde{\mathcal{M}}^-$ implies that $\frac{\partial^2 g_{\bar{u}}(s,t)}{\partial s^2}|_{(1,1)} < 0$ and $\frac{\partial^2 g_{\bar{u}}(s,t)}{\partial t^2}|_{(1,1)} < 0$. Therefore, there is a constant $\sigma \in (0,1)$ such that

$$0 < m_0 := \max_{\partial D} I(\rho\bar{u}^+ + \tau\bar{u}^-) < \widetilde{m},$$

$$\max_{(\rho,\tau)\in D}\frac{\partial^2 g_{\rho\bar{u}^+ + \tau\bar{u}^-}(s,t)}{\partial s^2}|_{(1,1)} < 0 \ and\ \max_{(\rho,\tau)\in D}\frac{\partial^2 g_{\rho\bar{u}^+ + \tau\bar{u}^-}(s,t)}{\partial t^2}|_{(1,1)} < 0 \tag{3.4}$$

where $D := [1-\sigma, 1+\sigma] \times [1-\sigma, 1+\sigma]$.

Supposed by contradiction that $I^{'}(\bar{u}) \neq 0$, we can find $\rho > 0$ and $\delta > 0$ to guarantee that

$$||I'(u)|| \geq \rho \ \ for\ all\ u \in E\ with\ ||u - \bar{u}|| < 3\delta.$$

In order to apply the deformation lemma ([39, Theorem 2.3]), let

$$\varepsilon := min\{\frac{\widetilde{m} - m_0}{3}, \frac{\rho\delta}{8}\}\ and\ S_\delta := \{u \in E: ||u - \bar{u}|| \leq \delta\}.$$

Based on the above notations, there is a deformation $\eta \in C([0,1] \times E, E)$ such that

- $\eta(r, v) = v$, if $r = 0$ or $v \notin I^{-1}([\widetilde{m} - 2\varepsilon, \widetilde{m} + 2\varepsilon] \cap S_{3\delta}$;
- $I(\eta(r, v)) \leq I(v)$ for all $v \in E$ and $r \in [0,1]$;
- $I(\eta(r, v)) < \widetilde{m}$ for any $v \in S_{2\delta}$ with $I(v) \leq \widetilde{m}$ and $r \in (0,1]$.

In what follows, for the convenience, let $h: D \to E$ be the function defined by

$$h(\rho, \tau) := \rho\bar{u}^+ + \tau\bar{u}^-, \forall(\rho, \tau) \in D.$$

Evidently, by Lemma 3.3 and (ii), for any $r \in [0,1]$, we get

$$\max_{\{(\rho,\tau)\in D: h(\rho,\tau)\notin S_{2\delta}\}} I(\eta(r, h(\rho, \tau))) \leq \max_{\{(\rho,\tau)\in D: h(\rho,\tau)\notin S_{2\delta}\}} I\big(h(\rho, \tau)\big) < \widetilde{m}.$$

Furthermore, we deduce from Lemma 3.3 and (iii) that

$$\max_{\{(\rho,\tau)\in D: h(\rho,\tau)\in S_{2\delta}\}} I(\eta(r, h(\rho, \tau))) < \widetilde{m}, \forall r \in [0,1].$$

Therefore, we see that

$$\max_{(\rho,\tau)\in D} I(\eta(r, h(\rho, \tau))) < \widetilde{m}, \forall r \in [0,1]. \tag{3.5}$$

Meanwhile, (3.4) and the continuity of η ensure the existence of $r_0 \in (0,1]$ small enough such that

$$\max_{(\rho,\tau)\in D} \frac{\partial^2 g_{\eta(r_0,h(\rho,\tau))}(s,t)}{\partial t^2}|_{(1,1)} < 0\ \ and\ \max_{(\rho,\tau)\in D} \frac{\partial^2 g_{\eta(r_0,h(\rho,\tau))}(s,t)}{\partial t^2}|_{(1,1)} < 0 \tag{3.6}$$

Thus, on account of (3.5) and (3.6), to finish the proof, it is sufficient to show that $\eta(r_0, h(D)) \cap \widetilde{\mathcal{M}}^- \neq \emptyset$. To this end, let us define $\phi_1, \phi_2: D \to R^2$ as

$$\phi_1(\rho, \tau) := (\langle I'(\rho\bar{u}^+ + \tau\bar{u}^-), \rho\bar{u}^+\rangle, \langle I'(\rho\bar{u}^+ + \tau\bar{u}^-), \tau\bar{u}^-\rangle)$$

and

$$\phi_2(\rho, \tau) := (\langle I'(\eta(r_0, h(\rho, \tau))), \eta^+(r_0, h(\rho, \tau))\rangle, \langle I'(\eta(r_0, h(\rho, \tau))), \eta^-(r_0, h(\rho, \tau))\rangle),$$

Since $\bar{u} \in \widetilde{\mathcal{M}}^-$, it is evident to derive that

$$\frac{\partial^2 g_{\bar{u}}(s,t)}{\partial s^2}\frac{\partial^2 g_{\bar{u}}(s,t)}{\partial t^2} - \frac{\partial^2 g_{\bar{u}}(s,t)}{\partial t \partial s}\frac{\partial^2 g_{\bar{u}}(s,t)}{\partial s \partial t}|_{(1,1)} > 0.$$

which, together with Lemma 3.3, infers that $deg(\phi_1, D, 0) = 1$. On the other hand, owing to $\varepsilon \leq \frac{\widetilde{m}-m_0}{3}$, that is, $m_0 < \widetilde{m} - 3\varepsilon < \widetilde{m} - 2\varepsilon$, we know that $\eta(r, h(\rho, \tau)) = h(\rho, \tau)$ for any $r \in (0,1]$ and $(\rho, \tau) \in \partial D$, which gives that

$$\phi_1(\rho, \tau) = \phi_2(\rho, \tau), \forall(\rho, \tau) \in \partial D.$$

Thus, the homotopy invariance property of the degree theory ([40, Proposition 1.34]) ensures that

$$deg(\phi_2, D, 0) = deg(\phi_1, D, 0) = 1.$$

As a result, there exists $(\rho_0, \tau_0) \in D$ such that $\phi_2(\rho_0, \tau_0) = 0$. From (3.6) and $\phi_2(\rho_0, \tau_0) = 0$, we deduce that $\eta(r_0, h(\rho_0, \tau_0)) \in \widetilde{\mathcal{M}}^-$. However, (3.5) implies this contradicts to the definition of $\widetilde{m}$. Therefore, $I'(\bar{u}) = 0$, namely, $\bar{u}$ is a sign-changing solution of equation (1.1).

4. Conclusion

The novelty of this paper is that when the nonlinear term of the fourth order elliptic equation is the combination of concave and convex functions, we obtain one sign-changing solution of the equation using the constraint variational method and quantitative deformation lemma. Since the usual projection property of $u^{\pm} \neq 0$ on $\mathcal{M}$ is no longer valid, it is unreasonable to choose $\mathcal{M}$ as the constraint set to find the existence of sign-changing solutions for our problem (1.1). In order to overcome this difficulty, we follow the technique utilized in [37] and construct one new non-empty closed subset $\widetilde{\mathcal{M}}^- \subset \mathcal{M}$. Then, it enables us to consider the corresponding infimum problem (1.9). With the help of the quantitative deformation lemma and the Brower degree theory, we obtain the minimizer of infimum problem (1.9), which is just one sign-changing solution of problem (1.1). These facts demonstrate that this method is effective in handling the existence of sign-changing solutions to local and nonlocal problems with concave and convex nonlinearities. Additionally, form the mathematical point of view, we believe that it is more challenging to search for additional hypothesis on $\kappa(x)$ to ensure multiple sign-changing solution.

Conflict of Interest

The authors declare there is no conflict of interest.

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