On Jump-Critical Ordered Sets with Jump Number Four

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Abstract: For an ordered set P and for a linear extension L of P, let $s(P,L)$ stand for the number of ordered pairs (x, y) of elements of P such that y is an immediate successor of x in L but y is not even above x in P. Put $s(P) = min\ \{s(P, L)$: L linear extension of $P\}$, the jump number of P. Call an ordered set P jump-critical if $s(P - \{x\}) < s(P)$ for any $x \in P$. We introduce some theorems about the jump-critical ordered sets with jump number four.

Keywords: Jump number, jump-critical ordered sets, tower poset.

1. INTRODUCTION

Let P be a poset and L be a linear extension of P. Every linear extension L of a finite ordered set P can be expressed as the linear sum $C_1 \oplus C_2 \oplus \ldots \oplus C_m$ of chains C_i of P so labeled that $sup_P\ C_i \nleqq inf_P\ C_{i+1}$ in P.

(*The linear sum* $A \oplus B$ of ordered sets A and B is the set $A \cup B$ ordered so that $a \leqq b$ provided that $a \in A$ and $b \in B$, or else, $a \leqq b$ in A or, $a \leqq b$ in B).

Let $C_i = \{a_i = a_{i1} < a_{i2} < \ldots < a_{ik_i} = b_i\}$. Then $b_i \nleqq a_{i+1}$ in P and such a pair (b_i, a_{i+1}) is called a *jump* (or *set up*) of the linear extension L, which is said to have m-1 jumps. We write $s\ (P, L) = m-1$. Note that a_{i+1} covers b_i in L, although $a_{i+1} \ngeqq b_i$ in P itself. We put $s(P) = min\ \{s\ (P,L)|\ L$ linear extension of $P\}$. This problem finds its practical settings too. Let the elements of P represent certain jobs to be performed one at a time by a single processor while the order of P imposes precedence constraints upon these jobs. Then an optimal linear extension of P is just a schedule of the jobs which minimizes the number of " *set up*" between unrelated jobs.

Observe that $s(P) \geq s(P - \{x\}) \geq s(P) - 1$ for any $x \in P$. A poset P is called jump-critical if $s\ (P – x) < s\ (P)$, for every element $x \in P$. If P is jump-critical with $s(P) = m$, then we say that P is m-jump-critical. It is easy to see that every ordered set P contains a jump-critical subset K with $s\ (P) = s\ (K)$. It may be that jump-critical ordered sets tell us much about the problem determining s(P) - even about constructing "optimal" linear extensions for P, that is, ones for which $s(P, L) = s(P)$. The ordered set illustrated in Figure **1** is jump-critical. Obviously, $s(P - \{a_{41}\}) < s(P))$. But to verify that $s(P - \{a_{31}\}) < 4$, for instance, requires a different chain decomposition: $P - \{a_{31}\} = C_2 \oplus C_4 \oplus C_5 \oplus \{a_{11} < a_{12} < a_{32}\}$. It is a good exercise to check all eight cases.

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The purpose of this paper is to stimulate activity on the jump number of an ordered set by recording several important examples. In section 2, we introduce some special methods to construct jump-critical ordered sets. In section 3, we introduce the complete lists of 1-jump-critical, 2-jump-critical, 3-jump-critical ordered sets and some theorems about 4-jump-critical ordered sets.

2. SPECIAL METHODS TO CONSTRUCT JUMP-CRITICAL ORDERED SETS

In this section we present special methods for constructing jump-critical posets. An *n*-element antichain is *jump-critical*. In fact, it is fairly obvious that the *disjoint sum* of jump-critical ordered sets is jump-critical. In addition, $s(P_1 + P_2) = s(P_1) + s(P_2) + 1$. It is equally obvious that the *linear sum* of jump-critical ordered sets is jump-critical. Also $s(P_1 \oplus P_2) = s\ (P_1) + s\ (P_2)$. These are special cases of a more general construction. Let P be an ordered set and each $x \in P$, let P_x be an ordered set. The *lexicographic sum* $\sum_{x \in P} P_x$ is the set $\bigcup_{x \in P} P_x$ ordered so that $u \leqq v$ if, for some $x \in P$, $u \in P_x$, $v \in P_x$ and $u \leqq v$ in P_x, or else, $u \in P_x$, $v \in P_y$, for some $x < y$ in P. It is implicit in M. Habib [5] that the lexicographic sum $\sum_{x \in P} P_x$ of critical ordered sets P_x is itself critical, as long as each $|P_x| > 2$. M. H. El-Zahar and I. Rival introduced a new method which gets jump-critical ordered sets by the theorem 1 [2].

Theorem 1: Let P_1 and P_2 be finite jump-critical ordered sets. Any ordered set obtained from P_1 and P_2 by gluing a maximal element of P_1 with a maximal

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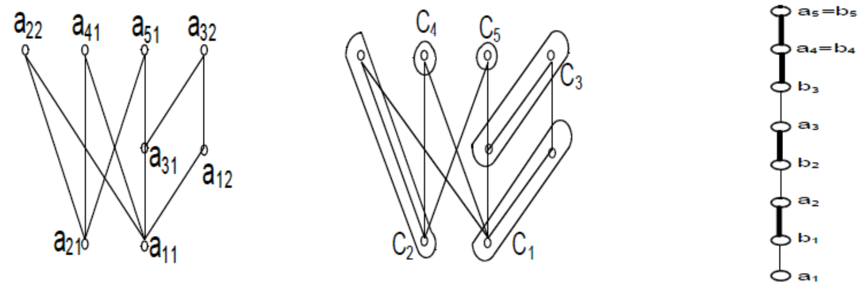


$s\,(P) = s\,(P, L) = 4$ $\quad P = \bigcup_{i=1}^{5} C_i$ $\quad L = \bigoplus_{i=1}^{5} C_i$

Figure 1:

element of P_2 is jump-critical and, in this case, the jump number is $s(P_1) + s(P_2)$. If | max P_1 | = | max P_2 | =2 then the ordered set obtained from P_1 and P_2 by gluing max P_1 with max P_2 is jump-critical and, in this case, the jump number is $s(P_1) + s(P_2)$ -1.

This gluing construction can be used to construct an example of jump-critical ordered set in which an "optimal" linear extension uses a long chain (see Figure **2**).

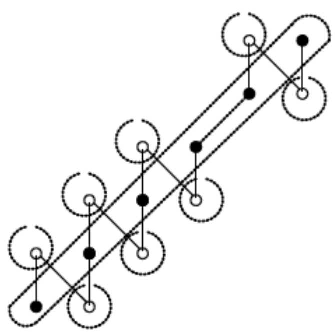


Figure 2:

There is an obvious question that arises for the second part of Theorem 1: does the gluing construction produce a jump-critical ordered set if there are more than two maximal elements? This question is open until now.

3. (1-4) JUMP-CRITICAL ORDERED SETS

In this section, we introduce the complete lists of 1-jump-critical, 2-jump-critical, 3-jump-critical ordered sets and some theorems about 4-jump-critical ordered sets.

Obviously, the only jump-critical ordered sets *P* with $s(P) = 0$ is the singleton. If $s(P) = 1$ then, of course, *P* must contain a noncomparable pair of elements. So, if *P* is jump-critical then *P* must be a two-element antichain. Suppose *P* is jump-critical and $s(P) = 2$. *P* may be a three-element antichain. The only other possibility is that *P* is the " four-cycle", as showed in Figure **3**. Thus, either $P \cong 1 + 1 + 1$ or $P \cong (1 + 1) \oplus (1 + 1)$.

M. H. El-Zahar and I. Rival [2] introduced the complete list of the jump-critical ordered sets with jump number three which has fourteen jump-critical ordered sets. These are, up duality, the ordered sets illustrated in Figure **4**.

Let *P* be a finite ordered set. For an element *a* in *P* put $D(a) = \{x \in P \mid x \leqq a\}$, a down set in *P*, $U(a) = \{x \in P \mid x \geqq a\}$, an upper set in *P*. Following M. H. El-Zahar and J. H. Schmerl [3] call the element *a* accessible in *P* if $D(a)$ is a chain in *P*. For instance, each minimal element of *P* is accessible. Let $w(P)$ stand for the width of *P*, the size of a maximum-sized antichain. According to Dilworth's chain decomposition theorem (1), *P* is the (disjoint) union of $w(P)$ chains. For maximum-sized antichains *A*, *B* in *P* we write $A \leqq B$ whenever for $a \in A$ there is $b \in B$ satisfying $a \leqq b$. (It follows, in this case that, for each $b \in B$ there is $a \in A$ satisfying $a \leqq b$, too). In this way the set of maximum-sized antichains of *P* is ordered: there is greatest (highest) antichain and a least (lowest) antichain. As matter of fact, the set of maximum-sized antichains is a distributive lattice in which $A \vee B = \max(A \cup B)$ and $A \wedge B = \min(A \cup B)$ (R. P. Dilworth [1]). A *tower* of height *k* (or *k-tower*) is a linear sum of *k-comparable* elements [4]. Obviously, a *k-tower* is *k*-critical with width two.

Theorem 2. Let *P* be a *k-jump-critical* ordered set with width 2 where *k* >1. Then *P* is a *k-tower*.

Proof: We use induction on *k*. For *k* = 2, the only poset which satisfies the criteria of the theorem is the 4-alternating-cycle $2 \oplus 2$. Thus, the result is true for *k* =

The only jump-critical ordered set with jump number one, the two-element antichain.

The two jump-critical ordered Sets with jump number two: the three- element antichain, and the four-cycle.

Figure 3:

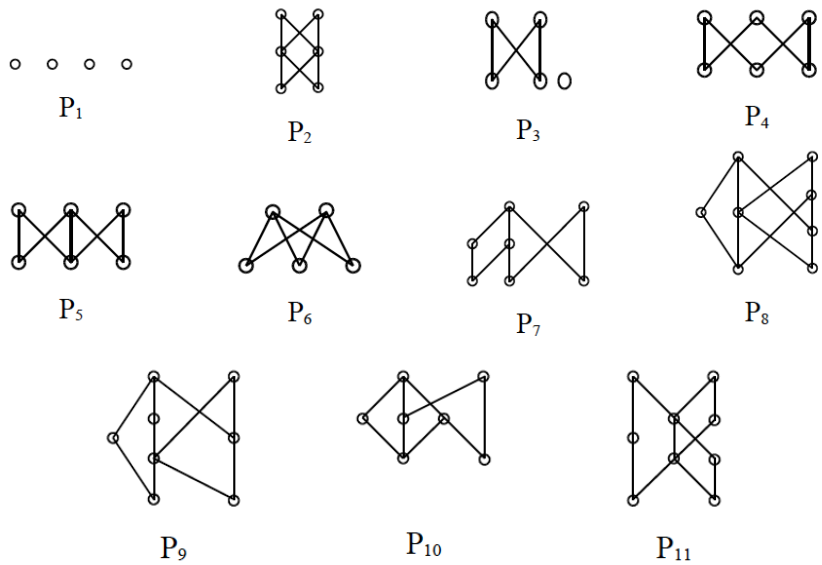


Figure 4:

2, and assume that it is true for jump-critical posets with jump-number less than k. Now we want to prove that it is true for jump-critical posets with jump-number k.

Since $w(P) = 2$ then it is the union of two chains C_1 and C_2. Put $x_i = \inf_p C_i$ for i = 1,2.

As P is jump-critical then $x_1 \neq x_2$. Let a_i be the maximal accessible element on C_i ; i = 1,2. See Figure **5**.

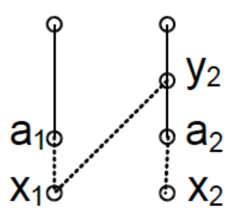


Figure 5:

We want to prove that $a_i = x_i$, $i = 1,2$. Suppose not, say $a_1 > x_1$. Put $P` = P - \{a_1\}$. As $P$ is jump-critical then $s(P`) = k - 1$. Let L be a linear extension of $P`$ with $k$-1 jumps, say $L = C_1` \oplus . \oplus C`_k$ where each $C`_i$ is a chain, i = 1, 2, k. If $x_1 \in C`_1$ then $C`_1 \cup \{a_1\}$ is also a chain. So, we can replace $C`_1$ on $L$ by $C`_1 \cup \{a_1\}$ which gives a linear extension of P with only k-1 jumps. This a contradiction. So, $x_1 \notin C`_1$ which implies that $C`_1 = x_2 \ldots a_2$. Now $x_1 \in C`_2$. If $C`_2 \cap C_2 = \varnothing$ then $C`_2 \cup \{a_1\}$ is a chain. Again, we can replace $C`_2$ by $C`_2 \cup \{a_1\}$ to get $s(P) = k$-1; a contradiction. Therefore $C`_2$ has the form $C`_2 = x_1 \ldots y_2 \ldots m$ where y_2 is the element that covers a_2 on C_2 and

$m = max\ C`_2$ is some element in $C_2$ (possibly $m = y_2$). Now we can replace $C`_1$ and $C`_2$ respectively by $C``_1$ and $C``_2$ where $C``_1 = x_1 \ldots a_1$ and $C``_2 = x_2 \ldots a_2\ y_2 \ldots m$. This gives a linear extension of P with only k-1 jumps which is a contradiction. We conclude that $a_1 = x_1$ and similarly $a_2 = x_2$. Now $P - \{x_1, x_2\}$ has jump number k-1 and, by induction, contains a (k-1) tower. This (k-1) tower together with $\{a_1, a_2\}$ forms a k-tower. This must be all of P. This completes the proof of the Theorem.

Theorem 3. There are precisely forty *jump-critical* ordered sets with four maximal elements and $s(P) = 4$. These are, up duality, the ordered sets illustrated in Figure **6**.

Proof of Theorem 3. It is straightforward, if

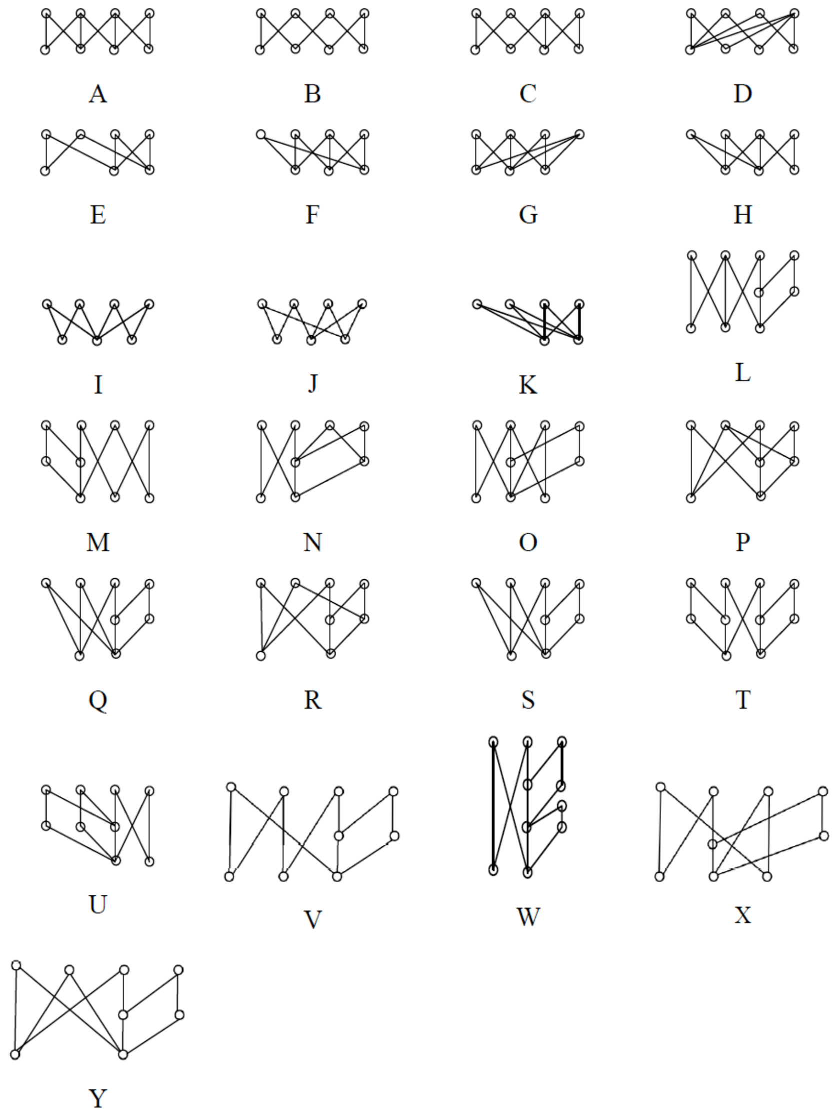


Figure 6:

somewhat laborious, to verify that each of the ordered sets illustrated in Figure **6** has jump number four, four maximal elements and that each is jump critical without isolated element.

Let P be 4-jump-critical and has four maximal elements (without isolated element). For contradiction, suppose that P contains no subset isomorphic to any of the posts illustrated in Figure 6.

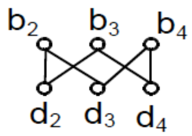
Since P is 4-jump-critical with $w(P) = 4$,then $P = C_1 \cup C_2 \cup C_3 \cup C_4$ (disjoint chains). Put $a_i = \inf_P C_i$ and $b_i = \sup_P C_i$ for i = 1, 2, 3, 4. Let us suppose that both $\{a_1, a_2, a_3, a_4\}$ is an antichain and $\{b_1, b_2, b_3, b_4\}$ is maximal elements antichain. If b_i 's is accessible, then $a_i \neq b_i$, $|D[b_i] \cap \{a_1, a_2, a_3, a_4\}| \geq 2$ and, dually, $|D[a_i] \cap \{b_1, b_2, b_3, b_4\}| \geq 2$. If follows that $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$ is isomorphic to A or B or C or D. Or that $\{a_1, a_2, a_3, b_1, b_2, b_3, b_4\}$ contains E (E^d) or $F(F^d)$ or $G(G^d)$ or $H(H^d)$ or $J(J^d)$.

Next, we handle the case $\{a_1, a_2, a_3, a_4\}$ is not antichain. Let $\{c_1, c_2, c_3, c_4\}$ be *infimum* of all four-element antichain in P.

One of c_i's must be less than one of b_i's, only, say $c_1 < b_1$, for otherwise the proper subsets $(\bigcup_{i=1}^{4} U[c_i])$ of P has jump number four. If $(P - U[c_1])$ contains four-element antichain, $\{x_1, x_2, x_3, x_4\}$ then c_1 must be comparable to one of these x_i's (say) x_1. But $x_1 > c_1$, since $x_1 \notin U(c_1)$ and if

$x_1 < c_1$ then $\{c_1, c_2, c_3, c_4\}$ is not the lowest four-element antichain in P. Therefore, $w\ (P - U[c_1]) = 3$

and we can assume that, $(P-U[c_1])$ = $C_2 \cup C_3 \cup C_4$ *so* that $U[c_1]$ = C_1. Let $\{d_2, d_3, d_4\}$ and $\{b_2, b_3, b_4\}$ be respectively, the lowest and highest, three-element antichain in $C_2 \cup C_3 \cup C_4$ where, say, $d_i, b_i \in C_i$ for both i = 2, 3, 4 since s ($C_2 \cup C_3 \cup C_4$ = 3) then $\{d_2, d_3, d_4, b_2, b_3, b_4\}$ is isomorphic to the following posets



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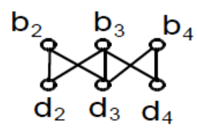


Neither b_i is above c_1. Also c_1 can not below d_i 's, otherwise c_1 < one of b_i 's only. Moreover $c_1 > d_2$ or $c_1 > d_3$ or $c_1 > d_4$. Otherwise c_1 is an isolated element in P. Therefore *min* (P) = *min* ($C_2 \cup C_3 \cup C_4$). For otherwise P would have a unique minimal element.

If $c_1 > d_2$, $c_1 > d_3$ and $c_1 > d_4$ then

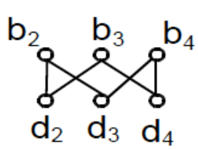
$\{c_1, d_2, d_3, d_4, b_2, b_3, b_4\} \cong E$ or $\{c_1, d_2, d_3, d_4, b_2, b_3, b_4\} \cong G$.

If c_1 > the two elements of $\{d_2, d_3, d_4\}$ then

$\{c_1, d_2, d_3, d_4, b_2, b_3, b_4\} \cong E$ or $\{c_1, d_2, d_3, d_4, b_2, b_3, b_4\} \cong H$.

We may then suppose that $c_1 > d_2$, $c_1 > d_3$ and $c_1 > d_4$. Since $b_1 = \sup_P C_1$, let us suppose that $b_1 > b_2$, $b_1 > b_3$ and $b_1 > b_4$ then there exists an element $d \in C_2 \cup C_3 \cup C_4$ such that $d \neq d_2$, $d < b_1$ and $c_1 \parallel d$.

Otherwise, c_1 is an accessible in the P^d, as $(P-U[c_1])$ has width three and jump number three so it must contain



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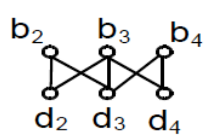
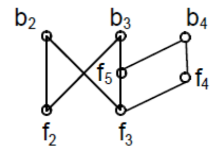


So that any of these Figures, with c_i is a subposet of P contains isolated element c_1. If b_1> one of $\{d_3, d_4\}$ then $\{b_1,b_2,b_3,b_4,d_2,d_3,d_4\} \cong F$ or $\cong H$. Otherwise

(i) $d_2 < d < b_2$, $d \parallel b_3$ and $d \parallel b_4$ or

(ii) $d_3 < d < b_5$, $d \parallel b_2$, $d \parallel b_4$ and $d \parallel b_1$

If (i) satisfies then $\{ b_1, c_1, d, b_2, b_3, b_4, d_2, d_3, d_4 \}$ is isomorphic to L, M, V or X; if (ii) satisfies then $\{ b_1, c_1, b_2, b_3, b_4, d_2, d_3, d_4, d\}$ is isomorphic to U. Now let $\{f_2, f_3\}$ and $\{b_2, b_3, b_4\}$ be, respectively, the lowest and height, two-element antichain and three-element antichain in $C_2 \cup C_3 \cup C_4$ where $f_i, b_i \in C_i$ for i = 2, 3, 4. Since $s(C_2 \cup C_3 \cup C_4)$ = 3 then P contains



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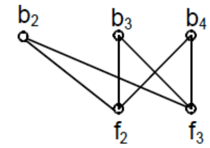


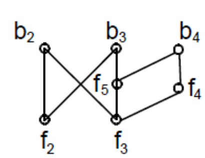
Neither b_i is above c_i. Also c_i can't below f_i 's otherwise, c_1 < one of b_i's only. Moreover $c_1 > f_2$ or $c_1 > f_3$ or $c_1 > f_4$ or $c_1 > f_5$ otherwise c_1 is isolated element in P. Therefore *min* (P) = *min* ($C_2 \cup C_3 \cup C_4$), for otherwise P would have a unique minimal element.

If $c_1 > f_2$ and $c_1 > f_3$, then $\{c_1, b_2, b_3, b_4, f_2, f_3\} \cong K$.

If $c_1 > f_2$ and $c_1 > f_4$ and $c_1 > f_5$; since $c_1 \parallel b_2$, $c_1 \parallel b_3$ and $c_1 \parallel b_4$ then $\{c_1, b_2, b_3, b_4, f_2, f_3, f_4, f_5\} \cong P$. If $c_1 > f_2$, $c_1 > f_4$ and $c_1 \parallel f_5$ since $c_1 \parallel b_2$, $c_1 \parallel b_3$ and $c_1 \parallel b_4$ then $\{c_1, b_2, b_3, b_4, f_2, f_3, f_4, f_5\} \cong R$, O or Y.

Now, if $c_1 > f_2$ and $c_1 > f_3$; since $b_1 = sup_P\, C_1$, let us suppose that $b_1 > b_2$, $b_1 > b_3$ and $b_1 > b_4$ then there is an element f $\in C_2 \cup C_3 \cup C_4$ such that $f \neq f_2$;

$f < b_1$ and c_1, f are incomparable, otherwise c_1 is an accessible in the P^d. As $(P-U[c_1])$ has width three and jump number three, it must contains



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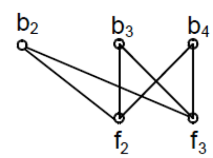

So that any of these Figures with c_1 is a subposet of P contains isolated element c_1. If $b_1 > f_3$ then $\{b_1, b_2, b_3, b_4, f_2, f_3\} \cong K$ and $\{b_1, b_2, b_3, b_4, f_2, f_3, f_4, f_5\} \cong S$ otherwise $f_2 < f < b_2$ and $f \parallel b_3$ and $f \parallel b_4$ then $\{c_1, f, b_1, b_2, b_3, b_4, f_2, f_3, f_4, f_5\} \cong T$. If $f_3 \prec c_1$ or $f_4 \prec c_1$ or $f_5 \prec c_1$ and $c_1 > f_2$; since $f_5 \neq f_3$, $f_5 > f_3$ therefore $f_5 \parallel c_1$, otherwise $P-(\bigcup_{i=1}^{4} U[c_i])$ has jump number four. Then, if $b_1 > f_5$ then $\{b_1, b_2, b_3, b_4, c_1, f_2, f_3, f_4, f_5\} \cong W$, if $f_4 \prec c_1$, $b > f$ and $f > f_4$ then $\{b_1, b_2, b_3, b_4, f_2, f_3, f_4, f_5, f, c_1\} \cong V$, if $f_5 \prec c_1$, $c_1 > f_4$ and $c_1 > f_2$ then $\{ c_1, f_2, f_3, f_4, f_5, b_2, b_3, b_4\} \cong N$ and if $f_5 \prec c_1$, $f_5 \prec f$, $b_1 > f$ then $\{b_1, b_2, b_3, b_4, f_2, f_3, f_4, f_5, f, c_1\} \cong W$. Hence this theorem is proved.

CONCLUSION

In this paper, we introduced some theorems about 4-jump-critical ordered sets. In future, we can investigate the structure of m-jump-critical ordered sets to study the jump-number problem.

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