Spatial Behavior of the Steady State Vibrations in a Dual-Phase-Lag Rigid Conductor

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Abstract: This paper studies the spatial behavior of the steady state vibrations in a cylinder made of a dual-phase-lag anisotropic rigid conductor material. We analyze the influence of the lagging model upon the spatial behavior of the amplitude of vibration along the axis of the cylinder, providing the explicit expressions of the decay rate and of the corresponding critical frequency in terms of the coefficients of the considered constitutive equation (or delay times). In fact, for the amplitude of the harmonic vibrations we obtain some exponential decay estimates of Saint-Venant type, provided the frequency of vibration is lower than a critical value. This gives information on the thermal penetration depth of the steady state vibrations describing the heat affected zone. Illustrative examples are given for the class of lagging behavior models that are thermodynamically compatible.

Keywords: Dual-hase-lag heat conduction model, Steady state vibrations, Spatial behavior, Rigid conductor.

1. INTRODUCTION

The physical foundation and microscale heat-transfer mathematical models, describing the lagging response in times comparable to the phase lags characterizing the microstructural interactions, have been presented in a series of papers by Tzou [1-4]. The refined structure of the lagging response is depicted by means of the high-order effects in correlation with the heat-transfer models in micro/nanoscale (like the systems with multiple energy carriers, including the bioheat transfer and mass interdifussion) in [3]. It is discussed in [4] the concept of thermal penetration depth for transient processes which limits the heat-affected zone as a finite domain. The heat-affected zone grows in the transient process of heat transport. The way in which the heat-affected zone enlarges in the time-history, however, is an unknown to be determined. A study on this question can be found in the paper by Chiriţă [5].

Throughout this paper we assume that the supply terms are absent and, according with the lagging behavior models, we further consider the basic energy equation

$$-q_{i,i}(\mathbf{x},t) = a(\mathbf{x})\frac{\partial T}{\partial t}(\mathbf{x},t), \tag{1}$$

coupled with the following constitutive equation

$$\begin{aligned} a_0 q_i(\mathbf{x},t) + \quad & a_1\frac{\partial q_i}{\partial t}(\mathbf{x},t) + \ldots + a_n\frac{\partial^n q_i}{\partial t^n}(\mathbf{x},t) \\ = \quad & -k_{ij}(\mathbf{x})\left[b_0 T_{,j}(\mathbf{x},t) + b_1\frac{\partial T_{,j}}{\partial t}(\mathbf{x},t) + \ldots + b_m\frac{\partial^m T_{,j}}{\partial t^m}(\mathbf{x},t)\right]. \end{aligned} \tag{2}$$

Here q_i are the components of the heat flux vector, $T_{,i}$ represents the gradient of the temperature variation, a is the specific heat and k_{ij} is the conductivity tensor and a_0, a_1, ..., a_n and b_0, b_1, ..., b_m are real parameters. Concerning this last constitutive equation we note that when

$$\begin{aligned} & a_0 = 1, \quad a_1 = \frac{1}{1!}\tau_q, \quad \ldots, \quad a_n = \frac{1}{n!}\tau_q^n, \\ & b_0 = 1, \quad b_1 = \frac{1}{1!}\tau_T, \quad \ldots, \quad b_m = \frac{1}{m!}\tau_T^m, \end{aligned} \tag{3}$$

we recover the constitutive equation proposed by Tzou [1]–[4], namely,

$$\begin{aligned} & q_i(\mathbf{x},t) + \frac{1}{1!}\tau_q\frac{\partial q_i}{\partial t}(\mathbf{x},t) + \ldots + \frac{1}{n!}\tau_q^n\frac{\partial^n q_i}{\partial t^n}(\mathbf{x},t) \\ & = -k_{ij}(\mathbf{x})\left[T_{,j}(\mathbf{x},t) + \frac{1}{1!}\tau_T\frac{\partial T_{,j}}{\partial t}(\mathbf{x},t) + \ldots + \frac{1}{m!}\tau_T^m\frac{\partial^m T_{,j}}{\partial t^m}(\mathbf{x},t)\right], \end{aligned} \tag{4}$$

where $\tau_q \geq 0$ and $\tau_T \geq 0$ are the delay times.

The spatial behavior of the transient solutions of the mathematical model based on the equations (1) and (2) was studied by Chiriţă [5] and Chiriţă *et al*. [6], describing the depth of thermal penetration. In fact, it was established in [5] an influence domain result when

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$m = n \pm 1$, while for $m = n$ there have been established some exponential decay estimates of the Saint-Venant type. These results have been established without requiring any restriction on the delay times other than the positivity of the product between the coefficients of the time derivatives of greatest order. These results offer information on the depth of thermal penetration of a transient process in the dual-phase-lag model of heat transfer.

The present paper aims to provide, through the use of the lagging behavior model described by the constitutive equation (2), a qualitative mathematical analysis of the problem concerning the penetration depth of a time harmonic thermal signal in an anisotropic and inhomogeneous rigid conductor. Thus, we consider a cylindrical domain filled by an anisotropic and inhomogeneous lagging behavior rigid conductor material and we assume that its lower base is subjected to a specific harmonic in time vibration. As it is known (see e. g. [7]-[10]) this is able to prove a steady state solution. Some characteristics of this steady state solution can be investigated by studying the spatial behavior of its amplitude with respect to the distance to the excited base of the cylinder. Such an investigation seems to be particularly interesting in view of the study of the nonlinear effects of thermal lagging (in terms of the high-order effects of τ_T and τ_q) and also in view of the fact that (see [4], page 442) as the various orders of τ_q and τ_T are gradually taken into account, which may result from increasing the number of carriers corresponding to higher-order effects in τ_q and τ_T. Moreover, when thermal lagging behaviors (the diffusive behavior and the wave-like behavior) are activated the spatial decay and the critical frequency values are affected. This contribution aims therefore to give information on the behavior of the amplitude of the steady state vibration and implicitly on the thermal penetration depth within the cylinder. In this connection we are able to introduce an appropriate functional associated with the amplitude in concern and to establish some exponential decay estimates in terms of the amplitude vibration, provided the frequency of vibration is lower than a critical value. The decay rate and the critical frequency are explicitly presented for the lagging behavior model based on the constitutive equation (4) when the values of approximation orders (n,m) make it compatible with the Second Law of Thermodynamics.

We have to outline that the corresponding studies for the classical elastic and thermoelastic theories have been carried out in [7-13]. It is established there that, for appropriate values of the frequency of the harmonic excitation and by using some differential inequalities, the spatial decay of effects with distance from the excited end of the right cylinder is described by some exponential decay estimates of Saint Venant type.

2. STEADY-STATE HARMONIC VIBRATIONS

Throughout this paper we assume that the cylinder $C = D \times (0,L)$ is made of a rigid conductor material with the lagging behavior and we suppose that the conductivity tensor is positive definite. We denote by D_{x_3} the plane cross-section at distance x_3 from the base of the cylinder, assuming it sufficiently smooth to allow the application of the divergence theorem; moreover, with C_{x_3} we denote the portion of cylinder between the cross-sections D_{x_3} and D_L. The cylinder is free of heat supply and it is thermally insulated on its lateral surface and on the end situated in the plane $x_3 = L$. The cylinder is subjected to a harmonic perturbation on its base $x_3 = 0$ of the form

$$T(x_1,x_2,0,t) = h(x_1,x_2)e^{i\omega t}, \quad (x_1,x_2) \in D_0, \quad t > 0, \tag{5}$$

where $\omega > 0$ is the frequency of perturbation and $i = \sqrt{-1}$ is the imaginary unit and $h(x_1,x_2)$ is a prescribed smooth function. Then inside of cylinder C we will have the following harmonic vibration

$$\{T, q_r\}(\mathbf{x},t) = \{\theta, Q_r\}(\mathbf{x})e^{i\omega t}, \tag{6}$$

where the amplitude $\{\theta, Q_r\}$ of the vibration is a solution of the boundary value problem defined by the differential system

$$Q_{r,r} = -i\omega a\theta, \quad for \quad all \quad \mathbf{x} \in C, \tag{7}$$

$$P_{(n)}\left(i\omega\right)Q_r = -R_{(m)}\left(i\omega\right)k_{rs}\theta_{,s}, \quad for \quad all \quad \mathbf{x} \in \overline{C}, \tag{8}$$

with the boundary conditions

$$\theta(\mathbf{x}) = 0 \quad on \quad \left(\partial D_{x_3} \times (0,L)\right) \cup D_L, \tag{9}$$

and

$$\theta(x_1,x_2,0) = h(x_1,x_2), \quad (x_1,x_2) \in D_0. \tag{10}$$

In the above relations we have used the following notations

$$\begin{aligned} P_{(n)}\left(i\omega\right) &= a_0 + a_1\left(i\omega\right) + ... + a_n\left(i\omega\right)^n, \\ R_{(m)}\left(i\omega\right) &= b_0 + b_1\left(i\omega\right) + ... + b_m\left(i\omega\right)^m. \end{aligned} \tag{11}$$

We are interested into the spatial behavior of the amplitude $\{\theta, Q_r\}(\mathbf{x})$ of the harmonic in time vibration $\{T, q_r\}(\mathbf{x},t)$, as defined by (6), along the axis of the right cylinder.

3. Mathematical analysis of the spatial behavior of the amplitude $\{\theta, Q_r\}(\mathbf{x})$

In order to study the spatial behavior of the amplitude $\{\theta, Q_r\}(\mathbf{x})$ we introduce the following functional

$$M(x_3) = -\int_{D_{x_3}} \left[R_{(m)}(i\omega) k_{3s}\theta_{,s}\overline{\theta} + \overline{R_{(m)}(i\omega)} k_{3s}\overline{\theta}_{,s}\theta \right] da, \quad x_3 > 0, \tag{12}$$

where a superposed bar denotes complex conjugate. Further, we note that the relations (8) and (12) imply

$$M(x_3) = \int_{D_{x_3}} \left[P_{(n)}(i\omega) Q_3\overline{\theta} + \overline{P_{(n)}(i\omega)}\overline{Q}_3\theta \right] da, \quad x_3 > 0, \tag{13}$$

and hence, by means of the divergence theorem and the use of relations (7) and (9), we get

$$\begin{aligned} -\frac{dM}{dx_3}(x_3) &= i\omega\left[P_{(n)}(i\omega) - \overline{P_{(n)}(i\omega)} \right] \int_{D_{x_3}} a\theta\overline{\theta} da \\ &\quad - \int_{D_{x_3}} \left[P_{(n)}(i\omega) Q_r\overline{\theta}_{,r} + \overline{P_{(n)}(i\omega)}\overline{Q}_r\theta_{,r} \right] da. \end{aligned} \tag{14}$$

Moreover, by using the relation (8), we have

$$P_{(n)}(i\omega) Q_r\overline{\theta}_{,r} + \overline{P_{(n)}(i\omega)}\overline{Q}_r\theta_{,r} = -\left[R_{(m)}(i\omega) + \overline{R_{(m)}(i\omega)} \right] k_{rs}\theta_{,r}\overline{\theta}_{,s}, \tag{15}$$

so that, the relation (14) becomes

$$\begin{aligned} -\frac{dM}{dx_3}(x_3) &= i\omega\left[P_{(n)}(i\omega) - \overline{P_{(n)}(i\omega)} \right] \int_{D_{x_3}} a\theta\overline{\theta} da \\ &\quad + \left[R_{(m)}(i\omega) + \overline{R_{(m)}(i\omega)} \right] \int_{D_{x_3}} k_{rs}\theta_{,r}\overline{\theta}_{,s} da. \end{aligned} \tag{16}$$

In view of the relation (11), we have

$$\begin{aligned} & i\omega\left[P_{(n)}(i\omega) - \overline{P_{(n)}(i\omega)} \right] = -2\omega^2 \pi_{(n)}(\omega), \\ & R_{(m)}(i\omega) + \overline{R_{(m)}(i\omega)} = 2\rho_{(m)}(\omega), \end{aligned} \tag{17}$$

where

$$\begin{aligned} \pi_{(n)}(\omega) &= a_1 - a_3\omega^2 + a_5\omega^4 - a_7\omega^6 + ..., \\ \rho_{(m)}(\omega) &= b_0 - b_2\omega^2 + b_4\omega^4 - b_6\omega^6 + ... \end{aligned} \tag{18}$$

Therefore, we have

$$-\frac{dM}{dx_3}(x_3) = -2\omega^2 \pi_{(n)}(\omega) \int_{D_{x_3}} a\theta\overline{\theta} da + 2\rho_{(m)}(\omega) \int_{D_{x_3}} k_{rs}\theta_{,r}\overline{\theta}_{,s} da. \tag{19}$$

At this stage we assume that

$$b_0 > 0. \tag{20}$$

It can be easily seen that for appropriate small enough values of ω we can make positive the following quantity

$$\rho_{(m)}(\omega) = b_0 - b_2\omega^2 + b_4\omega^4 - b_6\omega^6 + ... \tag{21}$$

In what follows we denote by $\omega_0 > 0$ the greatest value of ω so that

$$\rho_{(m)}(\omega) > 0, \quad for \quad all \quad 0 < \omega < \omega_0. \tag{22}$$

Our analysis needs a discussion of the sign of the first integral term in the right hand side of the relation (19) and in this aim we consider the following cases: (i) $\pi_{(n)}(\omega) a \le 0$ for appropriate small enough values of ω and for all $\mathbf{x} \in \overline{C}$, and (ii) $\pi_{(n)}(\omega) a > 0$ for appropriate small enough values of ω and for all $\mathbf{x} \in \overline{C}$.

Let us first consider the case (i) and we denote by $\hat{\omega}_1$ the greatest value of ω so that

$$\pi_{(n)}(\omega) a \le 0, \quad for \quad all \quad 0 < \omega < \hat{\omega}_1, \quad \mathbf{x} \in \overline{C}. \tag{23}$$

Consequently, from the relation (19), we deduce that

$$-\frac{dM}{dx_3}(x_3) \ge 2\rho_{(m)}(\omega) \int_{D_{x_3}} k_{rs}\theta_{,r}\overline{\theta}_{,s} da \ge 0, \tag{24}$$

for all $0 < \omega < \min\{\omega_0, \hat{\omega}_1\}$.

Let us now consider the case (ii) and we denote by $\tilde{\omega}_1$ the greatest value of ω so that

$$\pi_{(n)}(\omega) a > 0, \quad for \quad all \quad 0 < \omega < \tilde{\omega}_1, \quad \mathbf{x} \in \overline{C}. \tag{25}$$

Furthermore, in view of the lateral boundary condition in (9), we have [14]

$$\int_{D_{x_3}} \theta_{,\alpha}\bar{\theta}_{,\alpha} da \geq \lambda \int_{D_{x_3}} \theta\bar{\theta} da, \quad (26)$$

where λ is the lowest eigenvalue in the two-dimensional clamped membrane problem for the cross section D_{x_3}.

Now, if we use the estimate (26) into relation (19), we deduce that

$$-\frac{dM}{dx_3}(x_3) \geq 2\left(\rho_{(m)}(\omega) - \frac{a_{max}}{\lambda\kappa_{min}}\omega^2\pi_{(n)}(\omega)\right)\int_{D_{x_3}} k_{rs}\theta_{,r}\bar{\theta}_{,s} da, \quad (27)$$

where $a_{max} = \sup_{\bar{C}}|a|$ and $\kappa_{min} = \inf_{\bar{C}} k_{(m)}$ and $k_{(m)}$ is the lowest eigenvalue of the conductivity tensor k_{rs}. Further, we note that the relation (18) implies

$$\rho_{(m)}(\omega) - \frac{a_{max}}{\lambda\kappa_{min}}\omega^2\pi_{(n)}(\omega) = b_0 - \left(b_2 + \frac{a_{max}}{\lambda\kappa_{min}}a_1\right)\omega^2 + \left(b_4 + \frac{a_{max}}{\lambda\kappa_{min}}a_3\right)\omega^4 - \left(b_6 + \frac{a_{max}}{\lambda\kappa_{min}}a_5\right)\omega^6 + \dots, \quad (28)$$

and hence we can choose $\omega = \tilde{\omega}_2$ so that

$$\rho_{(m)}(\omega) - \frac{a_{max}}{\lambda\kappa_{min}}\omega^2\pi_{(n)}(\omega) > 0 \quad for \quad all \quad 0 < \omega < \min\{\omega_0, \tilde{\omega}_1, \tilde{\omega}_2\}. \quad (29)$$

Concluding, for the case (ii) we have

$$-\frac{dM}{dx_3}(x_3) \geq 2\gamma(\omega)\int_{D_{x_3}} k_{rs}\theta_{,r}\bar{\theta}_{,s} da, \quad (30)$$

where

$$\gamma(\omega) = \rho_{(m)}(\omega) - \frac{a_{max}}{\lambda\kappa_{min}}\omega^2\pi_{(n)}(\omega) \quad for \quad 0 < \omega < \min\{\omega_0, \tilde{\omega}_1, \tilde{\omega}_2\}. \quad (31)$$

With such a choice for $\gamma(\omega)$ as in (31) and by an integration with respect to x_3 variable over $[x_3, L]$, and the use of the end boundary condition (9) into relations (24) and (30), we obtain

$$M(x_3) \geq 2\sigma(\omega)\int_{C_{x_3}} k_{rs}\theta_{,r}\bar{\theta}_{,s} dv \geq 0, \quad (32)$$

recalling that $C_{x_3} = D \times (x_3, L)$. In the above relation we have used

$$\sigma(\omega) = \rho_m(\omega) \quad for \quad all \quad 0 < \omega < \min\{\omega_0, \hat{\omega}_1\}, \quad (33)$$

for the case (i), while

$$\sigma(\omega) = \gamma(\omega) \quad for \quad all \quad 0 < \omega < \min\{\omega_0, \tilde{\omega}_1, \tilde{\omega}_2\}, \quad (34)$$

for the case (ii).

On the other side, by means of the Cauchy-Schwarz and the arithmetic-geometric mean inequalities, from the relation (13) we obtain

$$\left|M(x_3)\right| \leq \int_{D_{x_3}}\left[\varepsilon P_{(n)}(i\omega)Q_3\overline{P_{(n)}(i\omega)Q_3} + \frac{1}{\varepsilon}\theta\bar{\theta}\right] da, \quad (35)$$

for every $\varepsilon > 0$. Furthermore, by means of relation (8), we have

$$P_{(n)}(i\omega)Q_3\overline{P_{(n)}(i\omega)Q_3} = \left[R_{(m)}(i\omega)\overline{R_{(m)}(i\omega)}\right]\left(k_{3s}\theta_{,s}k_{3r}\bar{\theta}_{,r}\right) \leq \left[R_{(m)}(i\omega)\overline{R_{(m)}(i\omega)}\right]\left(k_{3s}k_{3s}\right)\left(\theta_{,r}\bar{\theta}_{,r}\right). \quad (36)$$

By combining the relations (26), (35) and (36), we obtain

$$\left|M(x_3)\right| \leq \int_{D_{x_3}}\left(\varepsilon\alpha + \frac{1}{\varepsilon\lambda}\right)\theta_{,r}\bar{\theta}_{,r} da, \quad (37)$$

where

$$\alpha(\omega) = R_{(m)}(i\omega)\overline{R_{(m)}(i\omega)} \quad \sup_{\bar{C}}\left(k_{3s}k_{3s}\right). \quad (38)$$

At this instant we choose the parameter ε so that

$$\varepsilon\alpha = \frac{1}{\varepsilon\lambda} \quad \Leftrightarrow \quad \varepsilon = \frac{1}{\sqrt{\lambda\alpha}} \quad (39)$$

and hence the relation (37) becomes

$$\left|M(x_3)\right| \leq \frac{2}{\kappa_{min}}\sqrt{\frac{\alpha}{\lambda}}\int_{D_{x_3}} k_{rs}\theta_{,r}\bar{\theta}_{,s} da, \quad for \ all \quad \omega > 0. \quad (40)$$

Concluding, from the relations (24), (30), (32) and (40), we obtain the first-order differential inequality

$$M(x_3) + \nu(\omega)\frac{dM}{dx_3} \leq 0, \quad for \quad all \quad x_3 \in (0, L), \quad (41)$$

where

$$\nu(\omega) = \frac{1}{\sigma(\omega)\kappa_{min}}\sqrt{\frac{\alpha(\omega)}{\lambda}}. \quad (42)$$

When integrated, the differential inequality (41) furnishes the estimate

$$0 \le M(x_3) \le M(0) e^{-\frac{x_3}{v(\omega)}}, \quad for \quad all \quad x_3 \in (0,L), \tag{43}$$

that expresses the exponential decay of the amplitude $\{\theta, Q_r\}$ with respect to the distance x_3 at the loaded base.

Thus, we have the following theorem.

Theorem 1 Suppose that k_{ij} is a positive definite tensor and the hypothesis (20) holds true. Then the spatial behavior of the amplitude $\{\theta, Q_r\}$ of the harmonic vibration (6) is described as follows:

(i) when the relation (23) is true we have the following exponential decay estimate

$$0 \le M(x_3) \le M(0) e^{-\frac{x_3}{v_1(\omega)}}, \quad for \quad all \quad x_3 \in (0,L), \tag{44}$$

for all ω lower than the critical value ω_{c1} as defined by

$$\omega_{c1} = \min\{\omega_0, \hat{\omega}_1\} \tag{45}$$

and

$$v_1(\omega) = \frac{1}{\rho_m(\omega)\kappa_{min}} \sqrt{\frac{\alpha(\omega)}{\lambda}}; \tag{46}$$

(ii) when the relation (25) holds true we have the following exponential decay estimate

$$0 \le M(x_3) \le M(0) e^{-\frac{x_3}{v_2(\omega)}}, \quad for \quad all \quad x_3 \in (0,L), \tag{47}$$

for all ω lower than the critical value ω_{c2} as defined by

$$\omega_{c2} = \min\{\omega_0, \tilde{\omega}_1, \tilde{\omega}_2\} \tag{48}$$

and

$$v_2(\omega) = \frac{1}{\gamma(\omega)\kappa_{min}} \sqrt{\frac{\alpha(\omega)}{\lambda}}. \tag{49}$$

Remark 1 We note that the hypothesis (20) is identically verified for the constitutive equation (4).

Remark 2 A less accurate description of the exponentially decay phenomenon in question may be obtained using the inequality (26) into relation (19) in order to deduce that

$$-\frac{dM}{dx_3}(x_3) \ge 2\phi(\omega) \int_{D_{x_3}} k_{rs} \theta_{,r} \bar{\theta}_{,s} da, \tag{50}$$

where

$$\phi(\omega) = \rho_{(m)}(\omega) - \frac{a_{max}}{\lambda \kappa_{min}} \omega^2 \left| \pi_{(n)}(\omega) \right|. \tag{51}$$

Further, we note that the relation (18) implies

$$\begin{aligned} \phi(\omega) \ge b_0 &- \left(b_2 + \frac{a_{max}}{\lambda\kappa_{min}} |a_1| \right) \omega^2 + \left(b_4 - \frac{a_{max}}{\lambda\kappa_{min}} |a_3| \right) \omega^4 \\ &- \left(b_6 + \frac{a_{max}}{\lambda\kappa_{min}} |a_5| \right) \omega^6 + \dots, \end{aligned} \tag{52}$$

and let us denote by ω^* the greatest value of ω so that

$$\phi(\omega) = \rho_{(m)}(\omega) - \frac{a_{max}}{\lambda \kappa_{min}} \omega^2 \left| \pi_{(n)}(\omega) \right| > 0 \quad for \ all \quad 0 < \omega < \omega^*. \tag{53}$$

When we couple the relations (40) and (50) we are lead to the counterpart of the estimates (44) and (47), namely

$$0 \le M(x_3) \le M(0) e^{-\frac{x_3}{v^*(\omega)}}, \quad for \quad all \quad x_3 \in (0,L), \tag{54}$$

where now the critical frequency is estimated as

$$\omega_c^* = \min\{\omega_0, \omega^*\}, \tag{55}$$

and

$$v^*(\omega) = \frac{1}{\phi(\omega)\kappa_{min}} \sqrt{\frac{\alpha(\omega)}{\lambda}}. \tag{56}$$

4 EXAMPLES

As an illustrative example we consider here the Tzou model described by the constitutive equation (4) and discussed within the thermomechanical compatibility analysis developed by Chiriţă *et al*. [15] for $n \le 4$ and $m \le 4$. So in what follows we will indicate the decay rate and the critical frequency for the constitutive equation (4) when $(n,m) \in \{(0,0),(1,1),(2,2),(3,3),(4,4),(1,0),(2,1),(3,2),(4,3)\}$. The values $(m,n) \in \{(0,1),(1,2),(2,3),(3,4)\}$

can be treated in a similar way by using the idea expressed in Chiriţă *et al*. [15].

(I) The case $(n,m)=(0,0)$: The decay estimates (44) and (54) hold true for any $\omega>0$ with

$$v_1(\omega)=v^*(\omega)=\frac{1}{\sqrt{\lambda\kappa_{min}}}\sqrt{\sup_{\bar{C}}\left(k_{3s}k_{3s}\right)}. \tag{57}$$

for indefinite sign of the constitutive coefficient a.

(II) The case $(n,m)=(1,0)$: When $a\le 0$ in $\bar{C}$ the decay estimate (44) holds true for any $\omega>0$ with

$$v_1(\omega)=\frac{1}{\sqrt{\lambda\kappa_{min}}}\sqrt{\sup_{\bar{C}}\left(k_{3s}k_{3s}\right)}, \tag{58}$$

while when $a>0$ in $\bar{C}$ the decay estimate (47) holds true with

$$v_2(\omega)=\frac{1}{\sqrt{\lambda\kappa_{min}}\left(1-\frac{\omega^2}{\omega_c^2}\right)}\sqrt{\sup_{\bar{C}}\left(k_{3s}k_{3s}\right)}, \tag{59}$$

and the critical frequency

$$\omega_{c2}=\sqrt{\frac{\lambda\kappa_{min}}{\tau_q a_{max}}}. \tag{60}$$

We have to outline that the estimate (54) holds true with $v^*(\omega)=v_2(\omega)$ and $\omega_c^*=\omega_{c2}$.

(III) The case $(n,m)\in\{(1,1),(2,1)\}$: When $a\le 0$, then the decay estimate (44) holds true for all $\omega>0$ with

$$v_1(\omega)=\frac{1}{\sqrt{\lambda\kappa_{min}}}\sqrt{\left(1+\tau_T^2\omega^2\right)\sup_{\bar{C}}\left(k_{3s}k_{3s}\right)}, \tag{61}$$

while when $a>0$ the decay estimate (47) holds true with

$$v_2(\omega)=\frac{1}{\sqrt{\lambda\kappa_{min}}\left(1-\frac{\omega^2}{\omega_c^2}\right)}\sqrt{\left(1+\tau_T^2\omega^2\right)\sup_{\bar{C}}\left(k_{3s}k_{3s}\right)}, \tag{62}$$

for the critical frequency

$$\omega_{c2}=\sqrt{\frac{\lambda\kappa_{min}}{\tau_q a_{max}}}. \tag{63}$$

We have to outline that the estimate (54) holds true with $v^*(\omega)=v_2(\omega)$ and $\omega_c^*=\omega_{c2}$.

(IV) The case $(n,m)=(2,2)$**:** In the case when $a\le 0$, the decay estimate (44) holds true with

$$v_1(\omega)=\frac{1}{\sqrt{\lambda\kappa_{min}}\left(1-\frac{\omega^2}{\omega_{c1}^2}\right)}\sqrt{\left(1+\frac{1}{4}\tau_T^4\omega^4\right)\sup_{\bar{C}}\left(k_{3s}k_{3s}\right)}, \tag{64}$$

for the critical frequency

$$\omega_{c1}=\frac{\sqrt{2}}{\tau_T}. \tag{65}$$

While when $a>0$ the decay estimate (47) holds true with

$$v_2(\omega)=\frac{1}{\sqrt{\lambda\kappa_{min}}\left(1-\frac{\omega^2}{\omega_{c2}^2}\right)}\sqrt{\left(1+\frac{1}{4}\tau_T^4\omega^4\right)\sup_{\bar{C}}\left(k_{3s}k_{3s}\right)}, \tag{66}$$

for the critical frequency

$$\omega_{c2}=\frac{1}{\sqrt{\frac{1}{2}\tau_T^2+\frac{\tau_q a_{max}}{\lambda\kappa_{min}}}}. \tag{67}$$

We have to outline that the estimate (54) holds true with $v^*(\omega)=v_2(\omega)$ and $\omega_c^*=\omega_{c2}$.

(V) The case $(n,m)=(3,2)$**:** When $a\le 0$ then the decay estimate (44) holds true with

$$v_1(\omega)=\frac{1}{\sqrt{\lambda\kappa_{min}}\left(1-\frac{1}{2}\tau_T^2\omega^2\right)}\sqrt{\left(1+\frac{1}{4}\tau_T^4\omega^4\right)\sup_{\bar{C}}\left(k_{3s}k_{3s}\right)}, \tag{68}$$

and the critical frequency is estimated as

$$\omega_{c1}=\min\left\{\frac{\sqrt{2}}{\tau_T},\frac{\sqrt{6}}{\tau_q}\right\}. \tag{69}$$

While when $a>0$ then the estimate (47) holds true with

$$v_2(\omega)=\frac{\sqrt{\left(1+\frac{1}{4}\tau_T^4\omega^4\right)\sup_{\bar{C}}\left(k_{3s}k_{3s}\right)}}{\sqrt{\lambda\kappa_{min}}\left[1-\left(\frac{1}{2}\tau_T^2+\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\right)\omega^2+\frac{a_{max}\tau_q^3}{6\lambda\kappa_{min}}\omega^4\right]}, \tag{70}$$

and the critical frequency is estimated as

$$\omega_{c2} = \min\left\{\frac{\sqrt{2}}{\tau_T}, \frac{\sqrt{6}}{\tau_q}, \tilde{\omega}_2\right\}, \quad (71)$$

where $\tilde{\omega}_2$ is the smallest positive root of the algebraic equation

$$1-\left(\frac{1}{2}\tau_T^2+\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\right)\omega^2+\frac{a_{max}\tau_q^3}{6\lambda\kappa_{min}}\omega^4=0. \quad (72)$$

The decay estimate (54) holds true with

$$v^*(\omega)=\frac{\sqrt{\left(1+\frac{1}{4}\tau_T^4\omega^4\right)\sup_C\left(k_{3s}k_{3s}\right)}}{\sqrt{\lambda}\kappa_{min}\left[1-\frac{1}{2}\tau_T^2\omega^2-\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\omega^2\left|1-\frac{1}{6}\tau_q^2\omega^2\right|\right]}, \quad (73)$$

and the critical frequency is estimated as

$$\omega_c^* = \min\left\{\frac{\sqrt{2}}{\tau_T}, \omega^*\right\}, \quad (74)$$

where ω^* is the smallest positive root of the algebraic equation

$$1-\frac{1}{2}\tau_T^2\omega^2-\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\omega^2\left|1-\frac{1}{6}\tau_q^2\omega^2\right|=0. \quad (75)$$

(VI) The case $(n,m)\in\{(3,3),(4,3)\}$**:** When $a\le 0$ then the decay estimate (44) holds true with

$$v_1(\omega)=\frac{\sqrt{\left[\left(1-\frac{1}{2}\tau_T^2\omega^2\right)^2+\omega^2\tau_T^2\left(1-\frac{1}{6}\tau_T^2\omega^2\right)^2\right]\sup_C\left(k_{3s}k_{3s}\right)}}{\sqrt{\lambda}\kappa_{min}\left(1-\frac{1}{2}\tau_T^2\omega^2\right)}, \quad (76)$$

and the critical frequency is estimated as

$$\omega_{c1} = \min\left\{\frac{\sqrt{2}}{\tau_T}, \frac{\sqrt{6}}{\tau_q}\right\}, \quad (77)$$

while when $a<0$ then the decay estimate (47) holds true with

$$v_2(\omega)=\frac{\sqrt{\left[\left(1-\frac{1}{2}\tau_T^2\omega^2\right)^2+\omega^2\tau_T^2\left(1-\frac{1}{6}\tau_T^2\omega^2\right)^2\right]\sup_C\left(k_{3s}k_{3s}\right)}}{\sqrt{\lambda}\kappa_{min}\left[1-\left(\frac{1}{2}\tau_T^2+\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\right)\omega^2+\frac{a_{max}\tau_q^3}{6\lambda\kappa_{min}}\omega^4\right]}, \quad (78)$$

and the critical frequency is estimated as

$$\omega_{c2} = \min\left\{\frac{\sqrt{2}}{\tau_T}, \frac{\sqrt{6}}{\tau_q}, \tilde{\omega}_2\right\}, \quad (79)$$

where $\tilde{\omega}_2$ is the smallest positive root of the algebraic equation

$$1-\left(\frac{1}{2}\tau_T^2+\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\right)\omega^2+\frac{a_{max}\tau_q^3}{6\lambda\kappa_{min}}\omega^4=0. \quad (80)$$

The decay estimate (54) holds true with

$$v^*(\omega)=\frac{\sqrt{\left[\left(1-\frac{1}{2}\tau_T^2\omega^2\right)^2+\omega^2\tau_T^2\left(1-\frac{1}{6}\tau_T^2\omega^2\right)^2\right]\sup_C\left(k_{3s}k_{3s}\right)}}{\sqrt{\lambda}\kappa_{min}\left[1-\frac{1}{2}\tau_T^2\omega^2-\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\omega^2\left|1-\frac{1}{6}\tau_q^2\omega^2\right|\right]}, \quad (81)$$

and the critical frequency is estimated as

$$\omega_c^* = \min\left\{\frac{\sqrt{2}}{\tau_T}, \omega^*\right\}, \quad (82)$$

where ω^* is the smallest positive root of the algebraic equation

$$1-\frac{1}{2}\tau_T^2\omega^2-\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\omega^2\left|1-\frac{1}{6}\tau_q^2\omega^2\right|=0. \quad (83)$$

(VII) The case $(n,m)=(4,4)$**:** In the case when $a\le 0$ then the decay estimate (44) holds true with

$$v_1(\omega)=\frac{\sqrt{\left[\left(1-\frac{1}{2}\tau_T^2\omega^2+\frac{1}{24}\tau_T^4\omega^4\right)^2+\omega^2\tau_T^2\left(1-\frac{1}{6}\tau_T^2\omega^2\right)^2\right]\sup_C\left(k_{3s}k_{3s}\right)}}{\sqrt{\lambda}\kappa_{min}\left(1-\frac{1}{2}\tau_T^2\omega^2+\frac{1}{24}\tau_T^4\omega^4\right)}, \quad (84)$$

and the critical frequency is estimated as

$$\omega_{c1} = \min\left\{\frac{\sqrt{6}}{\tau_q}, \frac{\sqrt{6-\sqrt{12}}}{\tau_T}\right\}. \quad (85)$$

When $a>0$ then the decay estimate (47) holds true with

$$v_2(\omega)=\frac{\sqrt{\left[\left(1-\frac{1}{2}\tau_T^2\omega^2+\frac{1}{24}\tau_T^4\omega^4\right)^2+\omega^2\tau_T^2\left(1-\frac{1}{6}\tau_T^2\omega^2\right)^2\right]\sup_C\left(k_{3s}k_{3s}\right)}}{\sqrt{\lambda}\kappa_{min}\left[1-\left(\frac{1}{2}\tau_T^2+\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\right)\omega^2+\left(\frac{1}{24}\tau_T^4+\frac{a_{max}\tau_q^3}{6\lambda\kappa_{min}}\right)\omega^4\right]} \quad (86)$$

and the critical frequency is estimated as

$$\omega_{c2} = \min\left\{\frac{\sqrt{6}}{\tau_q}, \frac{\sqrt{6-\sqrt{12}}}{\tau_T}, \tilde{\omega}_2\right\}, \quad (87)$$

where $\tilde{\omega}_2$ is the smallest positive root of the algebraic equation

$$1-\left(\frac{1}{2}\tau_T^2+\frac{a_{max}\tau_q}{\lambda\kappa_{min}}\right)\omega^2+\left(\frac{1}{24}\tau_T^4+\frac{a_{max}\tau_q^3}{6\lambda\kappa_{min}}\right)\omega^4=0. \quad (88)$$

Remark 3 It has to be outlined that the classical Fourier law of heat conduction (that is the case $(n,m)=(0,0)$) imposes no critical limit upon the frequency of vibration in concern. Instead, when the genuine lagging behavior is considered, as it can be seen from the above examples, a critical frequency there seems to be imposed in order to establish the exponential decay estimates.

CONCLUDING DISCUSSIONS

Theorem 1 (by the estimates (44) and (47)) and Remark 2 (by the estimate (54)) describe some Saint Venant effects for the steady state solution (6) of the general thermal lagging model based on the fundamental equations (1) and (2), provided the frequency is lower than a prescribed critical value as given, correspondingly, by the relations (45), (48) and (55) and under the only assumption that $b_0 > 0$. At first glance we can observe the existence of a critical frequency limiting the class of the steady state vibrations endeavoured with the Saint Venant effects, in contrast with the classical Fourier model of heat conduction where such limiting frequency is absent (see the item (I) of the above section).

On the other side the spatial behavior of the transient solutions, for the model in concern, was established by Chiriţă [5]. It was shown there that (a) for $n=m+1$ there was established an influence domain result (the corresponding constitutive equation describes the wave behavior), while (b) for $n=m$ there was established some exponential decay estimates of the Saint-Venant type (the corresponding constitutive equation describes the diffusive behavior), without any restriction on the coefficients of the constitutive equation (2) other than $a_n b_m > 0$. Instead for the steady state solutions the spatial behavior is established in the present paper under the only hypothesis that b_0 is positive.

Going further to the examples presented in the above section, we have to outline that our analysis points out the following aspects:

(i) the spatial behavior of the steady state solutions within the Fourier model is established without any restriction upon the sign of the coefficient a and without any critical frequency;

(ii) for the Maxwell-Cattaneo model (characterized by the delay time τ_q), the study of the spatial behavior of the steady state solutions introduces a critical frequency depending on the time delay when $a>0$, while when $a \le 0$ the existence of the critical frequency is superfluous;

(iii) by increasing the orders of the delay time τ_T into the constitutive equation (4) results in a decreasing in the spatial decaying rate;

(iv) by increasing the orders of the two delay times τ_T and τ_q into the constitutive equation (4) results in diminishing the values of the critical frequency for which the corresponding spatial decay is established.

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